Chern-Simons Functional

Gives a diagram-free "def'n" of the Jones polynomial.

Sources: – Witten's paper, gives an invariant of links in general 3-mfld.

– Atiyah's book, "Knots in physics"

§1 Connections + Curvature (Andrew Lobb).

Look at fibre bundle $F \hookrightarrow P \xrightarrow{\pi} M$ where F is a Lie group (hence P principal).

G – Lie group

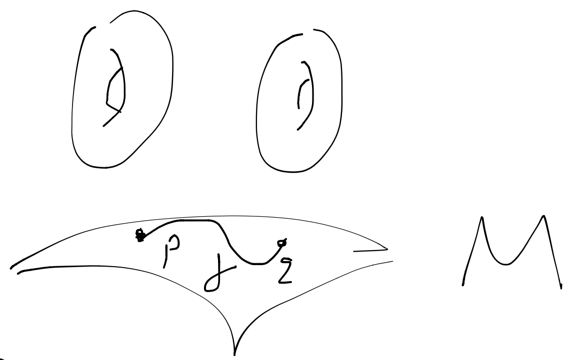
$\mathfrak{g}$ – Lie algebra $T_I G$ or left invariant vector fields on G.

Connection

Idea is to give a way to identify fibres with one another

e.g.

$G = T^2$

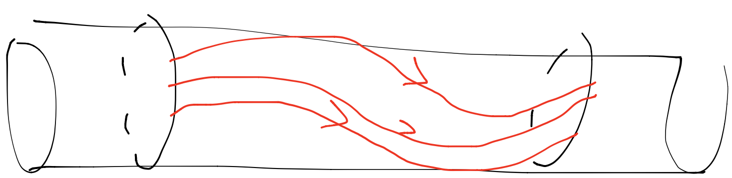


$\gamma : [0,1] \longrightarrow M$

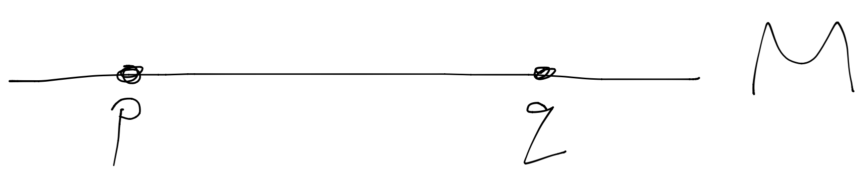
$\gamma(0) = p$

$\gamma(1) = q$

idea:



Define the horizontal paths γ.



If $u \in P$, then $T_u F \subseteq T_u P$

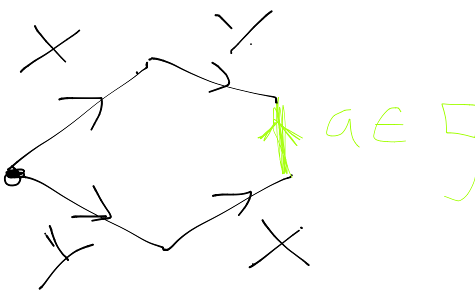
Def'n

A *connection* Γ is a choice for each $u \in P$ of a space $Q_u \subseteq T_u P$ such that $T_u P = \underbrace{Q_u}_{\text{horiz}} \oplus \underbrace{T_u F}_{\text{vert}}$.

Additionally, we require $Q_{ua} = (R_a)_* Q_u$ where we are using the right action of G on P; $P/G = M$.
This gives a way to identify fibres (parallel transport).

Curvature

Heuristic: this should measure the failure of parallel transport to commute infinitesimally (i.e. holonomy).
So we expect curvature to be a thing that eats $X, Y \in T_pM$ and excretes a Lie algebra element.



A connection Γ is equivalent to giving a Lie-algebra-valued 1-form ω on P satisfying some conditions:

(a) $\omega(A^*) = A$ $\qquad A \in \mathfrak{g}$, A^* vertical vector field on P.

(b) $(R_a)^* \omega = ad(a^{-1})\omega$ $\quad a \in G$, $ad(a^{-1}): \mathfrak{g} \to \mathfrak{g}$ where $Ad(a^{-1}): G \to G$, $g \mapsto a^{-1} g a$ and $ad(a^{-1})$ is differential.

Question How to see this is a splitting of

$$0 \to \text{vert} \to TP \to \text{horiz} \to 0$$

Suppose we have a rep $\rho: G \to GL(V)$ V a vector space.

Def'n A pseudotensorial form of degree r on P of type (ρ, V) is a V-valued r-form $\varphi \in \Omega^r(P, V \times P)$
s.t. $R_a^* \varphi = \rho(a^{-1})\varphi$. φ is tensorial iff it is horizontal: $\varphi(v_1, v_2, \dots, v_r) = 0$ if one v_i is vert.

Ex ρ_0 triv rep $\rho_0(x) = id_V$. Then (check) φ is invariant under pushing φ around by R_a^* (any a).
hence $\varphi = \pi^*(\varphi_M)$, some pullback.

Ex In general, a tensorial form φ of type (ρ, V) can be regarded as an r-form on M with values in $\Gamma(E)$ where E is the V-bundle over M, given by ρ.

How? $\begin{matrix} u \in P \\ \downarrow \quad \downarrow \\ x \in M \end{matrix}$ $\quad v_1, \dots, v_r \in T_x M$ and $\tilde{\varphi}_x(v_1, \dots, v_r) = u(\varphi(v_1, \dots, v_r))$ where recall the associated bundle construction defines for each $u \in P$ a vector space map $u: V \to E_x$

1. If φ is pseudo tensorial then φh is tensorial where h is projection to horizontal bundle
2. $d\varphi$ is also pseudo tensorial

CURVATURE

$D\varphi := (d\varphi)h$, $F := D\omega$ a tensorial 2-form of type $(ad, \mathfrak{g})$.

Remark If Ω^i is tensorial i-forms of type $(ad, \mathfrak{g})$

$$\Omega^0 \xrightarrow{D} \Omega^1 \xrightarrow{D} \Omega^2 \to \dots \quad \text{and } D^2 \text{ "=" } F \text{ somehow.}$$