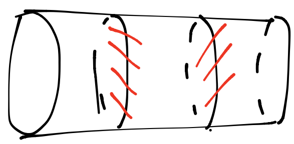
Holonomy

Recap

Principal bundle



$U(1)$-bundle over line

A connection is a collection of horizontal tangent spaces
$Q_u \subseteq T_u P \quad u \in P.$

Equivalently Γ is equiv to a pseudotensorial 1-form of type $(ad, \mathfrak{g})$ where $\mathfrak{g} = T_{id} G$.
Recall $D: \Omega^i \to \Omega^{i+1}$ for tensorial forms of type (ρ, V) where $\rho: G \to GL(V)$ a rep.

Cartan's Structure Equation:

$$d\omega = -\tfrac{1}{2}[\omega, \omega] + \Omega$$

i.e. $d\omega(X,Y) = -\frac{1}{2}[\omega(X), \omega(Y)] + \Omega(X,Y)$ for $X, Y \in T_u P$ $u \in P$.

Proof Check 3 cases:

1. X, Y horiz
2. X, Y vert
3. X hor. Y vert.

1: $\omega(X) = \omega(Y) = 0$. And $d\omega(X,Y) = \Omega(X,Y)$ ✓

2: Assume $X = A^*$, $Y = B^*$ vector fields corr to $A, B \in \mathfrak{g}$, const.

$$2 d\omega(A^*, B^*) = A^*(\omega(B^*)) - B^*(\omega(A^*)) - \omega[A^*, B^*]$$

$$\cancel{A^*(B)} - \cancel{B^*(A)} - [A, B]$$

and note $\Omega(X,Y) = 0$

③:

Let X be a horiz vector field, $Y = A^*$.

$\Omega(X, A^*) = 0$, $\omega(X) = 0$, $2d\omega(X, A^*) = A^*\omega(X) - X\omega(A^*) - \omega[X, A^*]$

$= 0 - 0 - \omega[X, A^*]$.

Will show $[X, A^*]$ horiz. But $-[X, A^*] = \mathcal{L}_{A^*}X$ (Lie deriv).

The 1-param family of diffeo of P induced by A^* is R_{a_t}, $a_t \in G$. (Check!)

Lie deriv is $\lim_{t\to 0} \frac{1}{t}(R_{a_t}X - X)$, a limit of horiz things. $\square$

<u>Cor</u> $\omega[X, Y] = -2\Omega(X, Y)$.

<u>Holonomy</u>

$G \circlearrowright P \xrightarrow{\pi} M$ with conn Γ. $x \in M$

$C(x)$ = loop space at x

$\cup\!|$

$C^\circ(x)$ = contractible-loop space. at x.

$\tau \in C(x)$ induces $\pi^{-1}(x) \circlearrowleft \cong$ and the collection of all such isos give the holonomy groups $\Phi(x), \Phi^\circ(x)$. A choice of identity point $u \in \pi^{-1}(x)$ gives a subgroup $\Phi(x) \cong \Phi(u) \subseteq G$. and differerent choice of u gives conjugate subgroup.

If $x, y \in M$ and $\pi_1(M) = \{1\}$ then $\Phi(x) \cong \Phi(y)$.

FACTS: • $\Phi^\circ(u)$ is a connected Lie subgroup of G.

• $\Phi^\circ(u) \trianglelefteq \Phi(u)$, and $\Phi(u)/\Phi^\circ(u)$ is countable.

<u>Reducing</u> (structure group, connection)

$$\begin{array}{ccccc} G' & \hookrightarrow & P' & \longrightarrow & M \\ & & \downarrow f & & \| \\ G & \hookrightarrow & P & \longrightarrow & M \end{array}$$

Also $\exists\, f: G' \to G$ s.t. $f(R_{g'}u') = R_{f(g')} f(u')$

$\underline{\text{e.g.}}$ $U(1) \overset{i}{\hookrightarrow} U(2)$ and $G' = U(1)$, $G = U(2)$

$\underline{\text{Th'm}}$ If we have a conn Γ' on P' then there is a unique conn Γ on P s.t. $f: P' \to P$ mapping $f_*: \text{hor}\, TP' \to \text{hor}\, TP$.

$\underline{\text{Def'n}}$ Γ is <u>reducible</u> to Γ' if

Let $u_0 \in P$, $P(u_0) \subseteq P$ defined to be the subset consisting of all points in P connected to u_0 by horizontal paths.

$\underline{\text{Th'm}}$
- $P(u_0)$ is a reduction of P with structure group $\Phi(u_0)$.
- The connection Γ is reducible to a conn on $P(u_0)$.

$\underline{\text{Th'm}}$ Let $u \in P$. The Lie alg of $\Phi(u)$ is subspace of $\mathfrak{g}$ spanned by $\Omega_v(X, Y)$ for all $v \in P(u)$, X, Y horiz vectors at v.