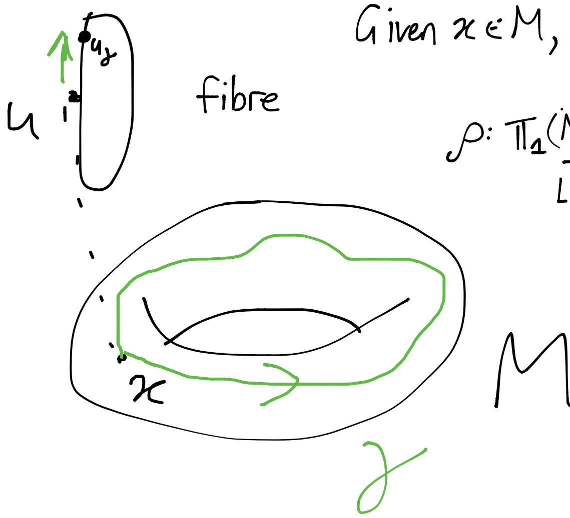
Flat Connections + Gauge

Canonical flat conn on $G \times M \to M$ is given by sections $\{g\} \times M$ for $g \in M$.
The connection form is the pullback of the canonical 1-form ω_G on G via $pr_1 : G \times M \to G$. ω_G is defined by $\omega_G : A^* \mapsto A$ for $A \in \mathfrak{g}$. The curvature is 0.

Using the holonomy theorems from last time:

$\Omega \equiv 0$ on M gives
- Lie algebra of $\Phi(u)$ and of $\Phi^0(u)$ are triv
- $\Phi^0(u) = \{e\}$.

The canonical flat conn can equiv be stated in terms of holonomy as the holonomy $\{e\}$ conn. So if we suppose $\Omega \equiv 0$ on $P \to M$, then if $D^n \subseteq M$ is a small n'hood then $P|_D \to D$ is the triv bundle and the connection is flat.



Given $x \in M$, $u \in \pi^{-1}(M)$, we get a homo

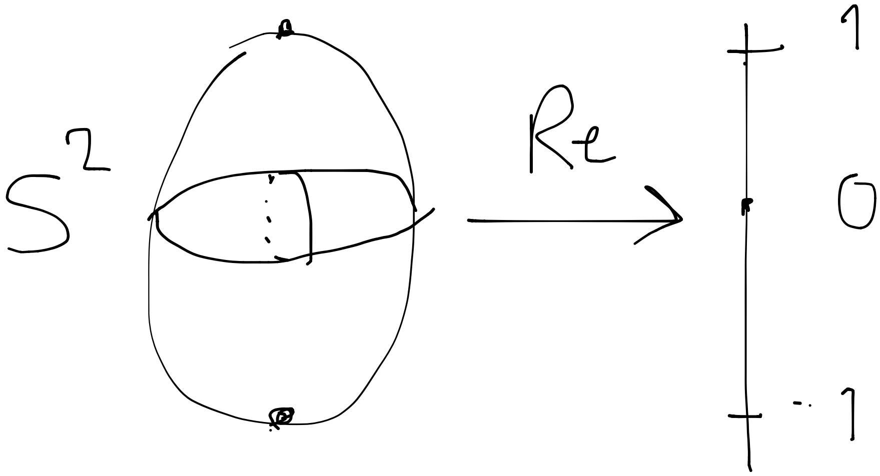
$$\rho: \pi_1(M) \longrightarrow G$$
$$[\gamma] \longmapsto g$$

Now recall there is the universal cover $\tilde{M} \to M$ and we can check $P = G \times_\rho \tilde{M}$.

Khovanov homology

Assigns to an oriented knot an abelian group.

$SU(2) \cong S^3$ = unit quaternions.



Each $\mathrm{Re}^{-1}(t)$ is a conjugacy class.

$K \subseteq S^3$. Think about flat $SU(2)$ connections on S^3 that are singular on the knot. This means

Fix the holonomy around "small" loops γ to lie in $Re^{-1}(0)$

e.g. $d\theta$ on $\mathbb{R}^2$ does not exist at the origin

The space of such connections is

$$\Sigma(K) = \{ \rho : \pi_1(S^3 \setminus K) \to SU(2) \mid \rho(m) \in Re^{-1}(0) \text{ for } m \text{ merid} \}$$

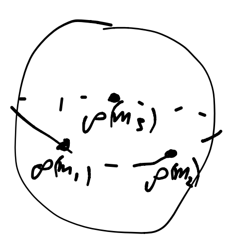
(note this has nothing to do with "smallness" but does rely on a basepoint).

e.g. $\Sigma(U) = Re^{-1}(0) = S^2$

e.g. $\Sigma(\text{trefoil}) = \{ \rho \mid \rho(m_i) \in Re^{-1}(0) \}$ where m_1, m_2, m_3 are the three meridional generators for $\pi_1(\text{tref})$.

So we get one bunch of ρ's such that $\rho(m_1) = \rho(m_2) = \rho(m_3)$
$\Rightarrow S^2$ (note this corr to $\pi_1 \to H_1$)

Theres another bunch where



$= Re^{-1}(0)$

this gives an $\mathbb{RP}^3$.

n.b. the coho of $\Sigma(K)$ gives Kh^* for certain classes of knots but there are also many counterexamples.

Fixing a gauge (Henry)

$G \to P \to M$ and $\omega \in \Omega_1(P, \mathfrak{g})$ (where $\mathfrak{g}$ is associated via the adjoint rep)

If $s: M \to P$ is a section of P then set $A = s^*\omega \in \Omega_1(M, \mathfrak{g})$.

Sections always exist locally so we can always do this locally.

Fix a tensorial form $\varphi \in \Omega_k(P, V)$. Set $\phi = s^*\varphi$. How do we recover φ?

set $x \in M, a \in G$ $\quad \varphi_{s(x)a} = a^{-1} \cdot \pi^* \phi$

To see this is right: RHS $= 0$ on vert vectors ✓

Next note $T_u P = \underset{\text{vert}}{Q_u} \oplus \underset{\text{hor}}{R_{a*} s_* T_{\pi(u)} M}$ where $s(x)a = u$.

Then if $x \in M, v \in T_x M$ (and recalling $R_a^* \varphi = \rho(a^{-1})\varphi$),

$a^{-1}\varphi(s_* v) = a^{-1} \cdot \phi(v) = a^{-1}\pi^*\phi(R_{a*} s_* v)$ ✓

If we use a different section $\tilde{s} = gs$ for $g: M \to G$ we get $\tilde{\phi} = g\phi$.

Now suppose we have a connection $\omega \in \Omega_1(P, \mathfrak{g})$. Take $A = s^*\omega$

$\omega_{s(x)a} = ad^{-1}(a^{-1})\pi^* A + a^*\Theta$ where Θ the canonical flat G connection over the triv n'hood determined by the section.

Now a change of section is a "gauge transformation"

$$\tilde{A} = Ad_g A + (g^{-1})^*\Theta.$$

<u>Remark</u> Physicists write holonomy around a loop C as

$$P\exp \int_C A \in G$$

"path ordered exponential".

Roughly, adding up all the connectiony bits over the loop, in the right order.

<u>e.g.</u> Non-triv flat A: $\quad M \times U(1) \ni (x, \lambda)$

Conn $A \in \Omega_1(M)$ wrt sections.

$\tilde{s}(x) = (x, \lambda(x))$ $\quad A \sim \tilde{A} = A - d\lambda$

$F = dA$, so flat conn $= \{\text{closed 1-forms}\} / \text{exact 1-forms.} = H^1_{dR}(M)$.

<u>Remark</u> Gauge group can either mean

$$G \quad \text{or} \quad M \to G.$$