Fixing a gauge (Henry)

$G \to P \to M$ and $\omega \in \Omega_1(P, \mathfrak{g})$ (where $\mathfrak{g}$ is associated via the adjoint rep)

If $s: M \to P$ is a section of P then set $A = s^*\omega \in \Omega_1(M, \mathfrak{g})$.

Sections always exist locally so we can always do this locally.

Fix a tensorial form $\varphi \in \Omega_k(P, V)$. Set $\phi = s^*\varphi$. How do we recover φ?

set $x \in M, a \in G$ $\quad \varphi_{s(x)a} = a^{-1} \cdot \pi^* \phi$

To see this is right: RHS $= 0$ on vert vectors ✓

Next note $T_u P = \underset{\text{vert}}{Q_u} \oplus \underset{\text{hor}}{R_{a*} s_* T_{\pi(u)} M}$ where $s(x)a = u$.

Then if $x \in M, v \in T_x M$ (and recalling $R_a^* \varphi = \rho(a^{-1})\varphi$),

$a^{-1}\varphi(s_* v) = a^{-1} \cdot \phi(v) = a^{-1}\pi^*\phi(R_{a*} s_* v)$ ✓

If we use a different section $\tilde{s} = gs$ for $g: M \to G$ we get $\tilde{\phi} = g\phi$.

Now suppose we have a connection $\omega \in \Omega_1(P, \mathfrak{g})$. Take $A = s^*\omega$

$\omega_{s(x)a} = ad^{-1}(a^{-1})\pi^* A + a^*\Theta$ where Θ the canonical flat G connection over the triv n'hood determined by the section.

Now a change of section is a "gauge transformation"

$$\tilde{A} = Ad_g A + (g^{-1})^*\Theta.$$

Remark Physicists write holonomy around a loop C as

$$P\exp \int_C A \in G$$

"path ordered exponential".

Roughly, adding up all the connectiony bits over the loop, in the right order.

e.g. Non-triv flat A: $\quad M \times U(1) \ni (x, \lambda)$

Conn $A \in \Omega_1(M)$ wrt sections.

$\tilde{s}(x) = (x, \lambda(x))$ $\quad A \sim \tilde{A} = A - d\lambda$

$F = dA$, so flat conn $= \{\text{closed 1-forms}\} / \text{exact 1-forms.} = H^1_{dR}(M)$.

<u>Remark</u> Gauge group can either mean

$$G \quad \text{or} \quad M \to G.$$