TQFT's – §2 Atiyah book

A TQFT in dim $d+1$ is a functor Z from $\mathcal{C}ob^{\circ}$, the category with objects compact, closed, smooth, oriented d-mflds and morphisms $(d+1)$-mflds (smooth oriented ...) with ∂, to the category of cx vector spaces (fin dim)

$Z : \mathcal{C}ob^{\circ} \longrightarrow Vect_{\mathbb{C}}$

n.b. we are denoting opp orientation by $-^{*}$.

Satisfying:

A1) Respects involution.

A2) Monoidal

A3) Functorial (redundant axiom as we said "functor"?)

A4) $Z(\emptyset) = \mathbb{C}$ ("——")

A5) $Z(\Sigma \times I)$ is identity. ("——")

Physical interpretation?

Consider a single particle moving in S^1

$t=0 \Rightarrow$ posn $q_I \in S^1$ But let $\mathcal{H}$ be the Hilbert space with basis given by the
$t=T \Rightarrow$ posn $q_F \in S^1$ points on S^1. Denote el't by $|q\rangle$ "ket" and the dual by $\langle q|$ "bra".

There is then a physical probability that a state q_I becomes q_F in time t, governed by some hamiltonian flow on S^1. $\hat{H}$ is induced on $\mathcal{H}$. This is written:

$$\left|\langle q_F | e^{-i\hat{H}t} | q_I \rangle\right|^2 \qquad (*)$$

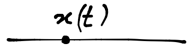
"Observables are unitary operators acting on $\mathcal{H}$"

e.g. $\hat{O}$ is an operator. If $|\psi\rangle$ is a eigen state then $\hat{O}|\psi\rangle = \psi|\psi\rangle$ now a scalar

Discussion about Hamiltonians:

H (classically) measures total energy. This is kinetic T plus potential V.

$$H = T + V = \frac{p^2}{2m} + V(x)$$

e.g. 1-dimension 

A Hamiltonian is a real-valued function on a symplectic manifold (spanned by (x,p)). This results is an energy minimising flow.

Quantisation means replacing classical quantities (pos, mom, ..) by operators (hermitian) on the Hilbert space of states. Now write:

$$\hat{H} = \frac{\hat{p}^2}{2m} + \hat{V}(\hat{x})$$

But operators may not commute (where once the quantities did)

e.g. $[\hat{x}, \hat{p}] = i h/2\pi$

so cannot measure position and momentum to arbitrary accuracy.

The Hilbert space has basis $\{|q_i\rangle\}$ where q_i is an eigenstate of the position operator i.e. $\hat{x}|q_i\rangle = q_i|q_i\rangle$. n.b. indexing set is in general uncountable (?).

!! To rigourously define all this one needs to take a discrete system (hence <u>countable</u> basis) and then take a "continuum limit", but physicists work already in the continuum limit because it's usually not a problem !!

Physical quantities are e.g. momentum:

- Change basis to e-values of $\hat{p}$ i.e. $|q\rangle = \sum_{\tilde{p}} c(\tilde{p})|\tilde{p}\rangle$ with $\langle \tilde{p}|q\rangle =: c(\tilde{p})$ and $\tilde{p}$ the e-vectors of $\hat{p}$

$$\langle q_I|\hat{p}|q_I\rangle = \sum_{\tilde{p},\tilde{p}'} \langle c(\tilde{p})\tilde{p}|\hat{p}|c(\tilde{p}')\tilde{p}'\rangle = \sum c(\tilde{p})^* c(\tilde{p}')\,\delta_{\tilde{p},\tilde{p}'}\,\tilde{p}'$$
$$= \sum_{\tilde{p}} |c(\tilde{p})|^2 \tilde{p} = \bar{p}$$

Then one can get other probabilistic quantities such as s.d. $\Delta p = \sqrt{\langle q|\hat{p}^2|q\rangle - \bar{p}^2}$

This leads to $\Delta x \Delta p \geq h/4\pi$ Heisenberg uncertainty principle.

Recall $(*)$ and now subdivide $T = \delta t \cdot N$ for some $N \in \mathbb{N}$:

$$\langle q_F | e^{-i\hat{H}T} | q_I \rangle = \langle q_F | e^{-i\hat{H}\delta t} e^{-i\hat{H}\delta t} \dots e^{-i\hat{H}\delta t} | q_I \rangle$$

Now the identity operator on $\mathcal{H}$ is $\int dq\, |q\rangle\langle q| = \mathbb{1}$ as

$$\int dq\, |q\rangle\langle q|x\rangle = \int dq\, |q\rangle \delta(q-x) = |x\rangle \quad \Rightarrow \text{basis is complete.}$$

Now insert this into our expression

$$\langle q_F | e^{-i\hat{H}\delta t} \int dq_1 |q_1\rangle\langle q_1| e^{-i\hat{H}\delta t} \int dq_2 |q_2\rangle\langle q_2| e^{-i\hat{H}\delta t} \dots e^{-i\hat{H}\delta t} | q_I \rangle$$

$$= \prod_{j=1}^{N-1} \int dq_j \, \langle q_F | e^{-i\hat{H}\delta t} | q_1 \rangle \dots \langle q_{N-1} | e^{-i\hat{H}\delta t} | q_I \rangle$$

Consider $\langle q_{j+1} | e^{-i\hat{H}\delta t} | q_j \rangle$

<u>example</u> Let $\hat{H} = \frac{\hat{p}^2}{2m} + \hat{V}(\hat{q})$ = Kinetic + potential

Let $V: \mathbb{C} \to \mathbb{C}$ be given by $\hat{V}(\hat{q})|q\rangle = V(q)|q\rangle$. Then

$$\langle q_{j+1} | e^{i\hat{H}\delta t} | q_j \rangle = e^{-i\delta t V(q_j)} \langle q_{j+1} | e^{-i\frac{\hat{p}^2}{2m}\delta t} | q_j \rangle \quad (\dagger)$$

cf. Baker Campbell Hausdorff eq.
we are ignoring commutation problems...?

The momentum basis $\{|p\rangle\}$ is also complete <u>and</u> for each p, q $\langle q | p \rangle = e^{ipq}$.

$$\text{Hence } (\dagger) = \int \frac{dp}{2\pi} e^{-i\delta t V(q_j)} \langle q_{j+1} | e^{-i\delta t \frac{\hat{p}^2}{2m}} | p \rangle\langle p | q_j \rangle$$

$$= \int \frac{dp}{2\pi} e^{-i\delta t \left(\frac{p^2}{2m} + V(q_j)\right)} \langle q_{j+1} | p \rangle\langle p | q_j \rangle$$

$$= \int \frac{dp}{2\pi} \exp\left(-i\delta t \left(\frac{p^2}{2m} + V(q_j)\right) + ip(q_{j+1} - q_j)\right)$$

Now recall $\int dx\, e^{-\frac{1}{2}ax^2} = \sqrt{\frac{2\pi}{a}}$

$$\Rightarrow (\dagger) = \exp\left(i\delta t\left(\frac{m}{2}\left(\frac{q_{j+1} - q_j}{\delta t}\right)^2 - V(q_j)\right)\right)\left(\frac{-im}{2\pi\delta t}\right)^{\frac{1}{2}}$$

Now sub $(\dagger)$ into $(*)$.

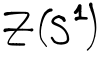
$$\langle q_F | e^{-i\hat{H}T} | q_I \rangle = \left(\frac{-im}{2\pi\delta t}\right)^{N/2} \prod_{j=1}^{N-1} \int dq_j \, e^{i\delta t \sum_{j=0}^{N-1} \left(\frac{m}{2}\left(q_{j+1}\right.\right.}$$

Back to TQFTs!

We will add another couple of axioms to make the theory unitary.

A6) All f.d. cx vector spaces carry a hermitian metric and morphisms $Z(Y^*)$ and $Z(Y)$ are adjoint.

Example Khovanov's TQFT

$\underline{d=1}$ $Z(S^1) := A = \mathbb{C}\langle v_+, v_- \rangle$, $Z($$) = \Delta : A \to A \otimes A$

$Z($$) = m : A \otimes A \to A$

where $A = \mathbb{C}[X] / X^2$ and $\begin{matrix} v_+ = 1 \\ v_- = X \end{matrix}$ with m, Δ the multiplication and co-multiplication resp.

Relative TQFTs for $d=2$

Now we want to fix a compact Lie group G and consider a triple (Y, L, Λ) where $L \subset Y^3$ is a properly embedded oriented 1-manifold and $\Lambda = (\lambda_1, \dots, \lambda_r)$ is a labelling of the r connected components of L by irreps of G. The TQFT axioms are modified in the obvious manner.

e.g.