Introduction to Witten's paper

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Quantum field theory and the Jones polynomial - Witten 1989. [W]

Sources: Much of this was lifted from Will Merry's "Quantum Mechanics via symplectic geometry".

Aim of talk - Say enough to state the goals of [W].

Plan
1. Recall what we did so far
2. Recall the Lagrangian/Hamiltonian pictures of classical physics
3. Recall the meaning of "observables" in classical + quantum physics
4. Discuss Chern-Simons theory

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(1) Andrew Lobb - Princ. G-bundles
- conn
- curvature
- holonomy

} gauge theory

Henry Mansfield - Gauge invariance

Sam Fearn - Quantisation

Will Rushworth - TQFTs.

To understand how to put these pieces together we need to talk more about classical physics.

② The path to understanding QFT (quantum field theory) for us will go from classical mechanics via two routes:

$$\begin{tikzcd}
\text{classical mechanics} \arrow[r, rightsquigarrow] \arrow[d, rightsquigarrow] & \text{quantum mechanics} \arrow[d, rightsquigarrow] \\
\text{field theory} \cup \text{gauge theory} \arrow[r, rightsquigarrow] & \text{QFT}
\end{tikzcd}$$

e.g. Yang-Mills + Chern-Simons

<u>Classical Mechanics</u> Suppose we wish to know how a particle moves on an n-manifold M. e.g. $M = \mathbb{R}^n$.

The position and velocity are in general independent variables $(q, v) \in TM$
$v \in T_q M$

<u>Def'n</u> The <u>configuration space</u> is the total space TM of the tangent bundle.
$\mathcal{M}$

An <u>action</u> is a function

$$S : \mathcal{M} \longrightarrow \mathbb{R}; \qquad S = \int_{t_0}^{t_1} \underbrace{L_t \, dt}_{}$$

S is called a "local function"

and $L : \mathcal{M} \times \mathbb{R} \longrightarrow \mathbb{R}$ is a choice of <u>Lagrangian</u>.
$(q, v, t) \longmapsto L_t(q, v)$.

The $\underline{\text{dynamics}}$ are then the paths $\mathbb{R} \xrightarrow{\gamma} M$ s.t. $(\gamma, \dot\gamma) \in \mathcal{M}$ is an extremum of S. } PRINC OF LEAST ACTION.

$\boxed{\text{Perturbation Analysis}} \Rightarrow$ The Euler-Lagrange equations, a 2nd order ODE in q.
+ Euler-Lagrange vector field determining flow on TM.

<u>Def'n</u> The space of solutions to the Euler-Lagrange equations is called covariant phase space. "Phase space" is sometimes reserved for a choice of parametrisation of covariant phase space.

<u>e.g.</u> In our case, a choice of position and momentum determines a unique solution. So the space of positions and momenta is phase space.

NOW HAMILTONIAN!

<u>Def'n</u> The Legendre transform is

$$\begin{aligned} \tau_L : TM \times \mathbb{R} &\longrightarrow T^*M \times \mathbb{R} \\ (q, v, t) &\longmapsto (q, D_v L(q,v,t), t) \end{aligned}$$

where $D_v L(q,v,t) := \sum_{i=1}^{n} \frac{\partial L_t}{\partial v_i}(q,v) dq_i$ (in a local basis).

This is the <u>momentum</u> of a velocity v at point q and time t.

A <u>Hamiltonian</u> is smooth f'n

$$H : T^*M \times \mathbb{R} \longrightarrow \mathbb{R}.$$

A Lagrangian determines a Hamiltonian by

$$H(\tau_L(q,v,t)) = D_v L(q,v,t)(v) - L(q,v,t).$$

n.b. we need to know τ_L is an iso. It is not always but let's ignore this now

The Legendre transform also determines the <u>Hamiltonian eq'ns</u> from the Euler-Lagrange equations. Moreover, the Hamiltonian vector field is a symplectic vector field on T^*M and this formalism can be exploited (of which more in subsequent lectures).

<u>Field theory</u>

<u>Def'n</u> Given M^n spacetime and X^m a target Riemannian mfld, a <u>field</u> is

$$\varphi : M \to X \quad \text{smooth}$$

<u>e.g.</u> $X = \mathbb{R} \Rightarrow$ scalar field

$X = TM$ and φ is a section $\Rightarrow$ vector field

~~Gauge f~~ Then we can do the Lagrangian/Hamiltonian thing again

e.g. $$S = \int_M L\left(\varphi, \nabla\varphi, \frac{\partial\varphi}{\partial t}, x \in M\right) dx \quad \text{etc.}$$

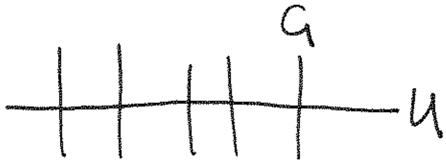
But we need gauge theory!

<u>SETUP</u> G a Lie group (conn, simple, compact, ...)

M a manifold (smooth, oriented, compact, has ∂, ...)

$G \to P \xrightarrow{\pi} M$ a G-princ bundle.

Briefly recall

For any x  s.t. 

open set $x \in U \subset M$ $= U \times G$

i.e.
$$\begin{tikzcd} \pi^{-1}(U) \arrow[r, "\cong"] \arrow[dr, "\pi"'] & G \times U \arrow[d, "pr_2"] \\ & U \end{tikzcd}$$

AND G acts on P s.t.

$y \in P_x = \pi^{-1}(x)$

$\Rightarrow$ $gy \in P_x$.

We have the adjoint rep $\mathfrak{g} = T_e G$

$$Ad : G \longrightarrow Aut(\mathfrak{g}); \quad g \longmapsto Ad_g \quad ,$$

where $\quad Ad_g : \mathfrak{g} \longrightarrow \mathfrak{g} \quad ; \quad a \longmapsto d\Psi_g|_e a \quad ,$

where $\quad \Psi_g : G \to G \quad ; \quad h \longmapsto hgh^{-1}.$

$$\begin{tikzcd} \text{PRINC G-BUNDLES} \arrow[r, bend left, "\text{Ad rep}"] & \text{VECTOR BUNDLES W/ STRUCTURE GROUP } G \arrow[l, bend left, "\text{associated structure bundle.}"] \end{tikzcd}$$

$P \longmapsto$ A vector bundle V ~~with~~ over M with $V_x = \mathfrak{g}$

Def'n A gauge field is a G-conn on $P \to M$

recall this means a splitting of $0 \to \overset{= \ker(d\pi)}{vert(TP)} \to TP \to horiz(TP) \to 0$

equiv. continuous choice of spaces $Q_u \subset TP$ $\forall u \in P$ s.t. $Q_u \oplus \underbrace{(\cancel{\pi^* TM})_u}_{\ker(d\pi)} \cong T_u P$.

+ there are compatibility conditions with G action.

<u>Def'n</u> A choice of section $s : U \subset M \underset{\pi}{\rightleftarrows} \pi^{-1}(U) \subset P$ for each local trivialisation n'hood U is called <u>fixing a gauge</u>.

$\Rightarrow$ We can pull stuff back to M, at least locally.

<u>Def'n</u> A ~~connection~~ gauge field determines a covariant derivative, setting

$$\Omega^r(P;\mathfrak{g}) = \Gamma(M, \Lambda^r T^*P \otimes_{Ad} \mathfrak{g}) \quad \text{smooth sections}$$

we have

$$\nabla : \Omega^0(P;\mathfrak{g}) \longrightarrow \Omega^1(P;\mathfrak{g})$$

linear s.t. if $f : P \to \mathbb{R}$ smooth and $s \in \Omega^q(P;\mathfrak{g})$ we have

$$\nabla(fs) = f\nabla s + df \otimes s. \qquad \text{(Liebniz rule).}$$

In particular, given a vector field X, we have over a local n'hood U.

$$\nabla_X s = \omega(X) s \qquad \text{for some } \omega \in \Omega^1(\pi^{-1}(U);\mathfrak{g}).$$

"connection potential"

$\omega(X)$ may not exist globally, but using our gauge fix, we may pull it back to

$$s^*\omega = A \in \Omega^1(U;\mathfrak{g}).$$

The co-ords of $\Omega^1(U;\mathfrak{g})$ are given by local tangent directions and basis of $\mathfrak{g}$

$$\Rightarrow \quad A = \sum A_i^a (dx_i \otimes a).$$

what phycisists call a connection.

Quantum Mechanics

	(hamiltonian) classical	$\rightsquigarrow$ quantum
ystem:	(M,ω) symp. mf'd.	$(\mathcal{H}, \langle\, , \rangle)$ cx seperable Hilbert space.
bservables:	(comm) The algebra $C^\infty(M)$	$\mathbb{L}^{sa}(\mathcal{H})$ the set of self-adjoint, unbounded linear operators $T: \mathcal{H} \longrightarrow \mathbb{C}$
States:	Don't worry about it for now	Positive, trace-class operators with trace 1.
easurement:		Don't worry about it for now.

<u>Def'n</u> Given a classical system (M,ω) a <u>quantisation</u> will be a choice of quantum system and an injection

$$\hat{\ } : C^\infty(M) \longleftrightarrow \mathbb{L}^{sa}(\mathcal{H}) ; \quad \Phi \longmapsto \hat{\Phi}$$

sending bounded f'n s to bounded operators + more conditions...

There are many measurements that can be taken of a quantum observable. For us we want <u>expectation</u>

$$\langle W \rangle_{\psi} := \langle \psi | W | \psi \rangle \quad \text{and} \quad \langle W \rangle := \langle W \rangle_{\substack{\text{vacuum} \\ \text{state}}}.$$

<u>Chern-Simons TQFT</u>

Fix some (poss non-abelian) $\overbrace{\text{conn, simple, compact } G, \text{ simply conn}}$. not necessary, but helpful.

M is a smooth oriented compact 3-mfld, $G \to P \to M$ a princ. G-bundle.

Configuration space $=$ space of gauge fields

<u>Remark</u> Let's assume $P = M \times G$ for now + fix a gauge $s : M \to M \times G$ $\Rightarrow$ gauge transformation group is $\mathrm{Map}(M, G)$.

Also $\Rightarrow$ configuration space $\cong \Omega^1(M; \mathfrak{g})$.

We now need to do two things:

- Set up dynamics, either via an action or by a Hamiltonian picture } We'll say something about this ~~now~~. later.
- Quantise the setup.
  - └ This will be next week's talk.

For each $k \in \mathbb{Z}$ there will be a different quantisation

$$Q_k : C^\infty(\overset{\text{phase space}}{\cancel{\text{gauge fields}}}) \longrightarrow \mathbb{L}^{sa}(\mathcal{H}_k)$$

$$W \longmapsto \hat{W}_k$$

We will be interested in certain observables called Wilson lines.

<u>Def'n</u> Let $K : S^1 \hookrightarrow M$ and λ be a rep of G. Fix a gauge field A.

$$W^{K,\lambda}(A) := \mathrm{Tr}_\lambda \left(\text{holonomy of } \cancel{\beta} \text{ around } K\right)$$

$$= \mathrm{Tr}_\lambda \, P\exp \int_K A_i \, dx_i$$

If $C = K_1 \sqcup K_2 \sqcup \ldots \sqcup K_r$ is a link then $\Lambda = (\lambda_1, \ldots, \lambda_r)$

$$W^{C,\Lambda}(A) := \prod_{j=1}^{r} W^{K_j, \lambda_j}(A).$$

<u>Th'm</u> (Witten) If $M = S^3$, $G = SU(2)$ and $\Lambda = (\lambda_1, \ldots, \lambda_r)$ where $\forall j$ λ_j = 2-d rep of $SU(2)$, then the quantum expectation

$\langle \hat{W}_k^{C,\Lambda} \rangle$ = ~~coeff of~~ value of the Jones polynomial ~~in quantum~~ evaluated at ~~degree~~ $q = \exp\left(\frac{2\pi i}{k+2}\right)$.