Geometric Quantisation 1

(Crash course in complex + symplectic geom.)

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Based on Alexei Kovalev's Complex manifold notes + Will Merry's Quantum Mechanics via Symplectic Geometry

Recap

To build Chern-Simons theory we pick G, M^3, $G \to P \to M$, fix a gauge $s: M \to P$, assume $P \cong M \times G$ then configuration space can be identified with $\Omega^1(M; \mathfrak{g})$, the $\mathfrak{g} = T_e G$ valued 1-forms on M.

To set up dynamics we pick an action $S: \Omega^1(M; \mathfrak{g}) \longrightarrow \mathbb{R}$.
We will discuss the action in a later talk.

We then solve the related Euler-Lagrange equations to obtain the phase space. We will discuss this in a later talk too.

Everything so far was "classical". We need to quantise the setup. There are several competing mathematical formalisms for this process (which tries to describe an actual physical phenomenon) The main 2:

- geometric quantisation
- perturbation quantisation

Witten uses geometric quantisation in his Jones polynomial paper, so we will start there. Hopefully there will be a later talk about the Reshetikhin-Turaev construction.

② Let's take this as our def'n on the quantisation we are trying to achieve:

<u>Def'n</u> Given a symplectic manifold (X, ω), a <u>quantisation</u> is a choice of complex seperable Hilbert space $(\mathcal{H}, \langle , \rangle)$ together with an injective map:

$$Q_\hbar : C^\infty(X, \mathbb{C}) \longrightarrow \mathbb{L}^{sa}(\mathcal{H})$$

self-adjoint linear operators $T : \mathcal{H} \to \mathcal{H}$.

such that:

(i) Bounded functions go to bounded functions

(ii) $$\lim_{\hbar \to 0} \frac{1}{2} Q_\hbar^{-1}\left(Q_\hbar(f) Q_\hbar(g) + Q_\hbar(g) Q_\hbar(f)\right) = fg \quad \forall f, g \in C^\infty(X, \mathbb{C})$$

(iii) $$\lim_{\hbar \to 0} Q_\hbar^{-1}\left(\frac{i}{\hbar}\left[Q_\hbar(f), Q_\hbar(g)\right]\right) = \{f, g\} \quad \forall f, g \in C^\infty_{bound}(X; \mathbb{C})$$

Where $\{f, g\}$ is the <u>Poisson bracket</u>.

<u>Remark</u> Sometimes there are additional axioms to make the setup correspond more precisely to the physics.

Strategy for geometric quantization:

Symplectic manifold (X, ω)

$\Big\downarrow$ prequantisation

Hermitian line bundle $\mathbb{C} \to L \to X$ with compatible conn ∇^L s.t. $\nabla^L \circ \nabla^L = -2\pi i \omega$.

$\Big\downarrow$

Prequantum Hilbert space $\mathcal{H}_0 = L^2(\Gamma(L))$ with $Q(f) := s \mapsto \nabla_{\Xi_f} s - 2\pi i f s$ where Ξ is Hamiltonian flow.

$\Big\downarrow$ polarisation

Use an almost complex structure to "cut $\mathcal{H}_0$ in half". to better represent the physics.

This lecture:

Say enough complex + symplectic geometry to explain the process above (in the next lecture).

Complex manifolds

Let X^{2n} be a smooth, closed, orientable manifold.

<u>Def'n</u> A complex structure on X is a system of charts $\varphi_\alpha : \mathcal{U}_\alpha \to \mathbb{C}^n$ ($\mathcal{U}_\alpha \subset X$) s.t. the transition functions $\varphi_\alpha \circ \varphi_\beta^{-1}$ are <u>holomorphic</u> where defined.

The underlying $2n$-dim <u>real</u> manifold is called $X_{\mathbb{R}}$.

In a local chart $\mathcal{U}$ we have a basis $z_j = x_j + iy_j$ $j=1,\dots,n$.

Then
$$T_p X_{\mathbb{R}} = \mathbb{R}\left\{ \frac{\partial}{\partial x_j}\Big|_p , \frac{\partial}{\partial y_j}\Big|_p \right\}$$

and
$$T_{\mathbb{C},p} X := T_p X_{\mathbb{R}} \otimes \mathbb{C} = \mathbb{C}\left\{ \frac{\partial}{\partial x_j}\Big|_p , \frac{\partial}{\partial y_j}\Big|_p \right\}$$

There is a linear map
$$J : T_p X_{\mathbb{R}} \longrightarrow T_p X_{\mathbb{R}}; \quad \frac{\partial}{\partial x} \mapsto \frac{\partial}{\partial y}, \quad \frac{\partial}{\partial y} \mapsto -\frac{\partial}{\partial x}$$

which extends linearly to
$$J : T_{\mathbb{C},p} X \longrightarrow T_{\mathbb{C},p} X$$

Note $J^2 = -\mathrm{id}$.

⑤ J defined on a real manifold may not extend to a bundle map. But on a complex manifold the holomorphic transition functions ensure it does.

Set:

$$T_p^{1,0}X := \{(J-i)v = 0\}$$ holomorphic tangent space

$$T_p^{0,1}X := \{(J+i)v = 0\}$$ anti-holomorphic tangent space.

$$T_p^{1,0}X = \mathbb{C}\left\{\frac{\partial}{\partial z_j}\Big|_p\right\}, \quad T_p^{0,1}X = \mathbb{C}\left\{\frac{\partial}{\partial \bar{z}_j}\Big|_p\right\}.$$

There is an iso:

$$\tau : TX_{\mathbb{R}} \longrightarrow T^{1,0}X$$
$$(p,v) \longmapsto (p, v - iJ(v)).$$

Def'n Similarly $J : (T_{\mathbb{C},p}X)^* \longrightarrow (T_{\mathbb{C},p}X)^*$ is defined and we have $T^*X^{1,0}$, $T^*X^{0,1}$, allowing us to describe forms of mixed type.

$$\Lambda^{p,q}X := \Lambda^p T^*X^{1,0} \wedge \Lambda^q T^*X^{0,1}$$

$$\Rightarrow \quad \Lambda^r T_{\mathbb{C}}^* X = \bigoplus_{p+q=r} \Lambda^{p,q}X$$

(6) And we have

$$\Omega^r_{\mathbb{C}}(X) = \Gamma(X, \Lambda^r T^*_{\mathbb{C}} X) \quad \text{the smooth sections}$$

$$\Omega^{p,q}(X) = \Gamma(X, \Lambda^{p,q} X)$$

The exterior derivative extends complex linearly

$$d_{\mathbb{C}} : \Omega^r_{\mathbb{C}}(X) \longrightarrow \Omega^{r+1}_{\mathbb{C}}(X)$$

But now we can decompose it into holomorphic and anti-holomorphic parts:

$$\left.\begin{array}{l} \partial : \Omega^{p,q}(X) \longrightarrow \Omega^{p+1,q}(X) \\ \bar{\partial} : \Omega^{p,q}(X) \longrightarrow \Omega^{p,q+1}(X) \end{array}\right\} \text{by the appropriate restriction and projection of } d.$$

<u>Lemma</u> $d|_{\Omega^{p,q}(X)} = \partial + \bar{\partial}$

Let's just check on 0-forms $f \in \Omega^0_{\mathbb{C}}(X) = C^\infty(X, \mathbb{C})$.

$$df = \frac{\partial f}{\partial z_j} dz_j + \frac{\partial f}{\partial \bar{z}_j} d\bar{z}_j \qquad \text{where } \{dz_j\} \text{ basis of } T^*X^{1,0}, \; \{d\bar{z}_j\} \text{ basis of } T^*X^{0,1}$$

$$\text{But indeed } \quad \frac{\partial f}{\partial z_j} dz_j = \partial f, \qquad \frac{\partial f}{\partial \bar{z}_j} d\bar{z}_j = \bar{\partial} f.$$

<u>FACT</u> f holo $\Rightarrow \bar{\partial} f = 0$.

<u>Def'n</u> The Dolbeault complex is

$$\cdots \xrightarrow{\bar{\partial}} \Omega^{p,q-1} X \xrightarrow{\bar{\partial}} \Omega^{p,q} X \xrightarrow{\bar{\partial}} \Omega^{p,q+1} X \xrightarrow{\bar{\partial}} \cdots$$

$\Rightarrow$ Dolbeault cohomology is $H^{p,q}(X) = H_q(\Omega^{p,*})$.

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Remark You might expect $H^r(X;\mathbb{C}) \cong \bigoplus_{p+q=r} H^{p,q}(X)$.

This is not in general the case. It is true when X is Kähler and we do a Hodge decomposition.

Complex vector bundles

Def'n A v.b. $\mathbb{C}^m \to E \xrightarrow{\pi} X$ is holomorphic if all its local trivializations are biholomorphisms

$$t : \pi^{-1}(U) \longrightarrow U \times \mathbb{C}^m \quad \text{biholo}$$

e.g. $T^{1,0}X$ is a holomorphic v.b.

Def'n A hermitian cx v.b. $(E \xrightarrow{\pi} X, h)$ is a cont. choice

$$h_p : E_p \times E_p \to \mathbb{C}. \qquad p \in X,\ E_p = \pi^{-1}(p).$$

of hermitian inner product (metric).

FACT Every complex v.b. admits a hermitian metric.

Given a cx v.b. $E \to X$, we can look at forms with values in E.

Def'n
$$\Omega^r_{\mathbb{C}}(X; E) = \Gamma(X, \Lambda^r T^*_{\mathbb{C}} X \otimes E)$$
$$\Omega^{p,q}(X; E) = \Gamma(X, \Lambda^{p,q} X \otimes E).$$

(8) A <u>covariant derivative</u> on E is a $\mathbb{C}$-linear map.

$$\nabla^E : \Omega^0_{\mathbb{C}}(X;E) \longrightarrow \Omega^1_{\mathbb{C}}(X;E)$$

satisfying the Leibniz property:

$$\nabla^E(fs) = df \otimes s + f\nabla^E s. \quad \forall \begin{array}{l} f \in C^\infty(X,\mathbb{C}) \\ s \in \Omega^0_{\mathbb{C}}(X,E). \end{array}$$

<u>FACT</u> Specifying a covariant derivative on E is equivalent to specifying a connection.

The <u>curvature</u> of the connection is

$$\nabla^E \circ \nabla^E : \Omega^0_{\mathbb{C}}(X;E) \longrightarrow \Omega^2_{\mathbb{C}}(X;E)$$

Complex line bundles

<u>Example</u> Let's classify all complex line bundles over a sphere $S^{N+1} = D^{N+1} \cup D^{N+1}$. Suppose $\mathbb{C} \to L \to S^{N+1}$ is such a bundle.

Then over the hemispheres L is $\cong D^{N+1} \times \mathbb{C} \xrightarrow{\text{TRIVIAL}} D^{N+1}$


$D^{N+1} \times \mathbb{C}$

How many ways are there to glue back together?


$D^{N+1} \times \mathbb{C}$

We must glue along the equator $S^N = \partial D^{N+1}$.

So at each point $p \in S^N$ we must give an automorphism

f had better be continuous if we want a v.b.

<u>FACT</u> If $f, g : S^N \to GL(1,\mathbb{C})$ homotopic the resultant v.b.s are iso.

$\therefore$ we only care about homotopy classes

$$[S^N, GL(1,\mathbb{C})] = [S^N, U(1)] \quad \left(\text{as } GL(1,\mathbb{C}) \simeq U(1)\right)$$

$$=: \pi_N(U(1)).$$

If $N > 1$ $\pi_N(S^1) = 0$. $\therefore$ all bundles iso to trivial bundle $\mathbb{C} \times S^{N+1}$

Set $N = 1$ for now.

$\pi_1(U(1)) \cong \mathbb{Z}$ (given by the winding #).

$\therefore$ There are $\mathbb{Z}$ such bundles.

<u>FACT</u> There is a space $BU(1)$ s.t. (in this case actually $BU(1) = \mathbb{C}P^\infty$)

$$[S^1, U(1)] = [S^2, BU(1)].$$

$f : S^2 \to BU(1)$ classifies the bundle L we get

The classification is given by $H^* BU(1) = \mathbb{Z}[c]$ where $c \in H^2(BU(1))$.

and $f^* c =: c_1(L) \in H^2(S^2; \mathbb{Z}) \cong \mathbb{Z}$.

<u>Th'm</u> For a general mfld M, the iso classes of cx line bundles are classified by $c_1(L) \in H^2(M; \mathbb{Z})$ which is defined by

$$c_1(L) := f^* c \quad \text{where } f : M \to \mathbb{C}P^\infty \text{ is the classifying}$$

map (i.e. $L \cong f^* EU(1)$ where $EU(1) \to BU(1)$ the tautological)

(10) <u>Def'n</u> (alternative) A complex v.b. $E \xrightarrow{\pi} X$ is <u>holomorphic</u> if there exists a <u>partial connection</u> (i.e. satisfies Leibniz and

$$\bar{\partial}^E : \Omega^{0,0}(X;E) \to \Omega^{0,1}(X;E) \quad)$$

s.t. $\bar{\partial}^E \circ \bar{\partial}^E = 0$. (n.b. $\bar{\partial}^E$ also called a Cauchy-Riemann operator).

We say a connection ∇^E is <u>compatible with the holomorphic structure</u> if in the decomposition $\nabla^E = (\nabla^E)^{1,0} + (\nabla^E)^{0,1}$ we have $\bar{\partial}^E = (\nabla^E)^{0,1}$.

A connection is <u>unitary</u> if it is compatible with the hermitian metric in the sense

$$d\langle s, s' \rangle = \langle \nabla^E s, s' \rangle + \langle s, \nabla^E s' \rangle$$

<u>Th'm</u> If $E \to X$ is a complex, holomorphic, hermitian v.b. then $\exists!$ connection ∇^E which is unitary + compatible with the holo structure. This is called the <u>Chern</u> connection.

The curvature 2-form $\Theta \in \Omega^2_{\mathbb{C}}(X)$ of the Chern connection is $\Theta \in \Omega^{1,1}(X)$ and $\bar{\partial}\theta = \Theta$ locally for some θ (where $\nabla^E s = \theta \cdot s$, $\theta \in \Omega^1_{\mathbb{C}}(X)$)

$\Rightarrow i\Theta$ is real.

<u>Th'm</u> If L is a complex line bundle then $c_1(L) = \left[\frac{i\Theta}{2\pi}\right] \in H^2_{dR}(X)$.

Th'm There is a natural bijection

$$\{\text{hermitian metrics } h \text{ on } T^{1,0}X\} \xleftrightarrow{1:1} \left\{\begin{array}{l}\text{Riemannian metrics } g \text{ on} \\ TX_{\mathbb{R}} \text{ s.t. } g(u,v) = g(Ju,Jv)\end{array}\right\}$$

Proof (Sketch) $\tau : T^{1,0}X \xleftarrow{\cong} TX_{\mathbb{R}}$ $\quad$ u,v vector fields in $TX_{\mathbb{R}}$

$\Rightarrow$ $\quad h \mapsto g$ where $g(u,v) := \frac{1}{2}\,\mathrm{Re}\{h(\tau(u),\tau(v))\}$. $\qquad (*)$

And, $\quad g \mapsto h$ by $h(u\otimes\lambda, v\otimes\mu) := \lambda\bar{\mu}\,g(u,v)$ $\quad \lambda,\mu \in \mathbb{C}$.

$\square$.

The other part of h is useful too.

Def'n The fundamental form $\omega \in \Omega^2(X_{\mathbb{R}})$ of h is

$$\omega(u,v) := \frac{1}{2}\,\mathrm{Im}\{h(\tau(u),\tau(v))\}$$

which can also be thought of as $\omega \in \Omega^2_{\mathbb{C}}(X)$ by linear extension.

Lemma (i) $\omega \in \Omega^{1,1}(X)$

(ii) For g as in $(*)$, $\quad \omega(u,v) = g(Ju,v)$

Proof (i) Suppose not. Then $\omega \in \Omega^{2,0}$ or $\Omega^{0,2}$. But if $\omega \in \Omega^{2,0}$ then for any $u,v \in T^{1,0}X$ we have $J(u), J(v) \in T^{0,1}X$ and

$\omega(u,v) = i\cdot -i\,\omega(u,v) = \omega(iu,iv) = \omega(Ju,Jv) = 0$. (as τ sends J to i)

But ω is non-degen on $X_{\mathbb{R}}$ $\quad$ #

similarly for $\Omega^{0,2}$

$\therefore \quad \omega \in \Omega^{1,1}X$

(ii) Calculate. $\quad\square$

(12) <u>Def'n</u> If $\eta \in \Omega^{1,1}X \cap \Omega^2(X_{\mathbb{R}})$ is s.t. $\forall v \neq 0 \in T^{1,0}X$ we have $-i\eta(v, J(v)) > 0$ then we say $\eta > 0$ and η is positive.

<u>Def'n</u> A positive, closed form $\omega \in \Omega^{1,1}X \cap \Omega^2(X_{\mathbb{R}})$ is called a Kähler structure on X. (X,ω) is called a Kähler manifold.

<u>Symplectic manifolds</u> Stop assuming X^{2n} is complex.

<u>Def'n</u> A closed, non-degenerate $\omega \in \Omega^2(X)$ is called a symplectic structure and (X,ω) is a symplectic manifold.

An <u>almost complex structure</u> on X is a bundle morphism $J: TX \to TX$ s.t. $J^2 = -\mathrm{id}$.

J is <u>integrable</u> if it arises as $J: \underset{\substack{\cong \\ TX_{\mathbb{R}}}}{T^{1,0}X} \to \underset{\substack{\cong \\ TX_{\mathbb{R}}}}{T^{1,0}X}$ from a cx structure.

<u>Remark</u> Integrability is related to the vanishing of the "Nijenhuis tensor".

<u>Def'n</u> A compatible triple $\{\omega, g, J\}$ is a symplectic form, Riemannian metric and almost complex structure s.t.

$$g(u,v) = \omega(u, Jv), \quad \omega(u,v) = g(Ju, v).$$

In fact, any 2 of the triple determine the third.

<u>Examples</u> A complex manifold X is almost complex.

If J in a compatible triple is integrable then (X,ω) is a Kähler manifold (where the symplectic form is extended complex linearly to $\omega \in \Omega^{1,1}X \cap \Omega^2(X_{\mathbb{R}})$.

Def'n The <u>(complex) Hamiltonian vector field</u> Ξ_f of $f \in C^\infty(X,\mathbb{C})$ is defined by

$$\omega(\Xi_f, -) = -df \qquad \text{(n.b. using non-degeneracy of } \omega\text{)}.$$

The Poisson bracket of $f,g \in C^\infty(X,\mathbb{C})$ is

$$\{f,g\} := df(\Xi_g) = \omega(\Xi_g, \Xi_f) = -\{g,f\} \in C^\infty(X,\mathbb{C}).$$

<u>Exercise</u> $\{fg,h\} = \{f,h\}g + \{g,h\}f$

<u>Lagrangians</u>

A <u>lagrangian subspace</u> of a symplectic vector space (V,ω) is a subspace

$$U \subset V \text{ s.t.}$$

U is maximally isotropic i.e. $\omega(u,v)=0 \ \forall v \in U \Rightarrow u \in U$.

$$U = U^\perp.$$

wrt the decomp $U \oplus U^* \cong V$ ω has matrix $\left(\begin{array}{c|c} 0 & * \\ \hline * & 0 \end{array}\right)$

We may choose a symplectic basis so that " $\left(\begin{array}{c|c} 0 & I \\ \hline -I & 0 \end{array}\right)$

Def'n Given (X^{2n}, ω) symplectic, a lagrangian submanifold is an embedding $i : Y^n \hookrightarrow X^{2n}$ s.t at each $p \in Y$ $di|_p : T_pY \hookrightarrow T_{i(p)}X$ is a lagrangian subspace.

Equiv. $i^*\omega \in \Omega^2(Y)$ vanishes.

Geometric Quantisation 2

Prequantisation

Let (X^{2n}, ω) be a symplectic manifold.

Assume $[\omega] \in \mathrm{im}\,(H^2(X;\mathbb{Z}) \to H^2(X;\mathbb{R}))$.

Then $[\omega]$ classifies a complex line bundle $\mathbb{C} \to L \to X$ with first Chern class ω. Moreover we may choose a hermitian metric h on L. From a previous theorem we know there exists a compatible connection ∇^L. We can also give L a holomorphic structure $\bar{\partial}^L := (1 + iJ)\nabla^L$.

Remark There is ambiguity in the above. The <u>ordinary differential cohomology</u> of X can be modelled by objects (L, h, ∇^L) as above in degree 2: $H^2_{\mathrm{diff}}(X)$. This fits into the SES's:

(a)
$$0 \to H^1(X; U(1)) \to H^2_{\mathrm{diff}}(X) \xrightarrow{F/2\pi i} \Omega^2_{\mathrm{int}}(X) \to 0$$
$$(L, h, \nabla^L) \longmapsto \omega$$

(b)
$$0 \to \Omega^1(X)/\Omega^1_{\mathrm{int}}(X) \to H^2_{\mathrm{diff}}(X) \xrightarrow{c} H^2(X;\mathbb{Z}) \to 0.$$

Given (L, h, ∇^L) as above consider $\Gamma_c(X, L)$, the compactly supported sections so there is defined
$s, s' \in$

$$\langle s, s' \rangle := \int_X h(s, s')\,\omega^n$$

n.b. ω^n is a volume form.

Set $\mathcal{H}_0$ to be the space of L^2 sections ~~of~~ $\Gamma_c(X, L)$ (i.e. the L^2-completion).

<u>Def'n</u> The cx Hilbert space $(\mathcal{H}_0, \langle , \rangle)$ is called the <u>prequantised Hilbert space</u> of (L, h, ∇^L).

The prequantisation is the assignment

$$Q : C^\infty(X, \mathbb{C}) \longrightarrow \mathrm{Hom}(\mathcal{H}_0, \mathcal{H}_0)$$

$$f \longmapsto Q(f) : s \longmapsto \cancel{\nabla^L_{\Xi_f} s - 2\pi i s f}$$

$$\cancel{\frac{1}{2\pi i} \nabla^L_{\Xi} s + fs}.$$

$$\nabla^L_{\Xi_f} s + 2\pi i f s.$$

<u>Lemma</u> - $[Q(f), Q(g)](s) = Q_{\{f,g\}}(s)$

<u>Proof</u> We have $\Xi_f = X$, $\Xi_g = Y$

$$[\nabla^L_{\Xi_f}, \nabla^L_{\Xi_g}](s) = (\nabla_X \nabla_Y - \nabla_Y \nabla_X)(s)$$

$$= F(X, Y)(s) + \nabla_{[X,Y]}(s)$$

for F the curvature 2-form.

$$= \nabla_{[X,Y]}(s) - 2\pi i\, \omega(X, Y)(s)$$

using $\nabla_{[X,Y]} = -\nabla_{\Xi_{\{f,g\}}}$

$$= -\nabla_{\Xi_{\{f,g\}}} s + 2\pi i \{f, g\} s \neq \cancel{Q(\{f,g\})(s)}$$

$$[Q(f), Q(g)] = [\nabla^L_X + 2\pi i f, \nabla^L_Y + 2\pi i g]$$

$$= [\nabla^L_X, \nabla^L_Y] + 2\pi i\left([\nabla^L_X, g] + [f, \nabla^L_Y]\right)$$ as f, g commute

But

$$[\nabla^L_X, g] + [f, \nabla^L_Y] = [\nabla^L_X, g] - [\nabla^L_Y, f]$$
$$= dg(X) - df(Y)$$
$$= \{g, f\} - \{f, g\}$$
$$= -2\{f, g\}.$$

For any Z, h

FACT $[\nabla^L_Z, h] = dh(Z)$ as ∇^L is compatible.

$$\Rightarrow [Q(f), Q(g)](s) = -\nabla_{\Xi_{\{f,g\}}} + 2\pi i\{f,g\} - 2\cdot 2\pi i\{f,g\}$$

$$= -Q(\{f,g\})(s).$$

□.

We now introduce the idea of a <u>level</u>. For any $m \in \mathbb{N}$, we have $m\omega$ an integral synmpleetic form and this will classify $L^{\otimes m} = L \otimes \dots \otimes L$. Denote by ∇^m our connection and $h^m = h \otimes \dots \otimes h$ the metric (so $\nabla^L = \nabla^1$). We can then define $\hbar = \frac{1}{m}$ and

$$Q_\hbar : f \longmapsto 2\pi i f + \nabla^m_{\Xi^m_f}$$

This now depends on a real # (like $\hbar = \frac{1}{m}$), obvs not continuously as $\{\frac{1}{m}\}$ is discrete!

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Example $X = \overset{\mathbb{R}^2}{\overset{\parallel}{T^*M}}$ $\overset{M=\mathbb{R}}{\cancel{(\text{some } M)}}$ and $\omega = dq \wedge dp$

Then $\mathbb{C} \to L \to X$ is trivial and sections $\Gamma(X, L)$ are $\mathrm{Map}(\mathbb{R}^2, \mathbb{C})$ But this Hilbert space doesn't make sense physically! It should be $\mathrm{Map}(\mathbb{R}, \mathbb{C})$ according to canonical quantisation.

Def'n Let (X, ω) be symplectic. A (complex) polarisation of (X, ω) is a choice at each $p \in X$ of a Lagrangian subspace $Y_p \subset T_pX \otimes \mathbb{C}$ s.t.

(1) If cx vector fields u_p, v_p lie in Y_p $\forall p$ then so does $[u, v]_p$

(2) $Y_p \cap \overline{Y}_p$ has constant dimension as p varies.

E.g. • (X, ω) is Kähler then set $Y_p := T_p^{1,0}X$

• $\begin{array}{c} X = T^*M \\ \downarrow \pi \\ M \end{array}$ then set $Y_{(q,p)} = \ker(d\pi : T_{\mathbb{C},(p,q)}X \to T_{\mathbb{C},q}M)$

the vertical polarisation.

We will now assume (X, ω) is Kähler and take the (canonical) polarisation $Y = T^{1,0}X$.

<u>Def'n</u> $\Gamma^{hol}(X, L^{\otimes m}) := \left\{ s \in \Gamma_c(X, L^{\otimes m}) \;\middle|\; \nabla_v s = 0 \;\; \forall v \in T^{0,1}X \right\}$

(more generally we would use $\overline{Y} \ni v$ and $\nabla_v s = 0$).

Alternatively recall our $\bar\partial^L = (1 + iJ)\nabla^L$. Then s is $\bar\partial^L s = 0$.

We now define the quantisation for $m = \frac{1}{\hbar}$, $\mathcal{H} = L^2(\Gamma^{hol}(X, L^{\otimes m}))$

$$\tilde{Q}_\hbar : C^\infty(X, \mathbb{C}) \longrightarrow \mathbb{L}(\mathcal{H})$$

This is the <u>Kostant-Souriau Quantisation</u>

by $\tilde{Q}_\hbar(f) = Q_\hbar(f)$ projected to $\mathcal{H}$.

<u>Remark</u>
- As X is compact, $\Gamma^{hol}(X, L^{\otimes m})$ is a finite dimensional subspace so this makes sense.
- In general one might take all the levels together
$$Q = (\tilde{Q}_1, \tilde{Q}_{\frac{1}{2}}, \tilde{Q}_{\frac{1}{3}}, \dots)$$
- The required asymptotic behaviour is not so easy to show.
- To make things match up even better with the physics there is a further stage called <u>metaplectic correction</u>. I can't see that Witten is doing this so we won't either.