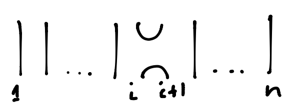
Algebraic Quantisation

Let TL be an algebra over $\mathbb{C}(q)$ generated by U_i, $i \in \mathbb{N}$ with

$$U_i^2 = (-q-q^{-1})U_i, \quad U_i U_{i\pm1} U_i = U_i, \quad U_i U_j = U_j U_i \text{ for } |i-j|>1$$

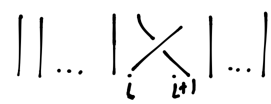
This gives the Temperley-Lieb algebra $TL_n = \langle U_1, \ldots, U_{n-1} \rangle$

We should think

$U_i =$  with $O = -q-q^{-1}$ and vertical stacking

Define <u>Jones trace</u> on TL $tr : TL \to \mathbb{C}(q)$ via

- $tr(ab) = tr(ba)$
- $tr(wU_n) = \frac{-1}{q+q^{-1}} tr(w)$ if $w \in TL_n$
- $tr(1) = 1$

Define $b_i = -q^{-1} - q^{-2}U_i$, $b_i^{-1} = -q - q^{-2}U_i$ $\qquad b_i =$ 

Given a braid b, define the <u>Jones polynomial</u>

$$V_{cl(b)}(q) = (-q-q^{-1})^n tr(b) \quad \text{where } n = \#\text{ strands in the braid.}$$

Hopf Algebras

An <u>algebra</u> A is a module over a commutative ring k with

$$m : A \otimes_k A \to A, \quad \eta : k \to A$$

s.t. $m(m \otimes 1) = m(1 \otimes m) : A^{\otimes 3} \to A$, $m(1 \otimes \eta) = m(\eta \otimes 1)$.

A Hopf algebra is an algebra A with alg hom

$$\Delta : A \to A \otimes_k A, \quad \varepsilon : A \to k \quad \text{and a lin map } S : A \to S$$

s.t. $(\Delta \otimes 1)\Delta = (1 \otimes \Delta)\Delta$, $(1 \otimes \varepsilon)\Delta = (\varepsilon \otimes 1)\Delta$, $S(ab) = S(a)S(b)$, $m(S \otimes 1)\Delta = m(1 \otimes S)\Delta = \eta\varepsilon$.

If M, N are modules over A can define a module over A $M \otimes_k N$ by

$$a \cdot (m \otimes n) = \Delta(a)(m \otimes n) = (a_1 \cdot m) \otimes (a_2 \cdot m) \quad \text{where } \Delta(a) = a_1 \otimes a_2.$$

Also if $M^* = \mathrm{Hom}_k(M, k)$ then M^* is an A-module via $(x \cdot f)(m) = f(S(x) \cdot m)$

We can then extend ev and coev to A-modules:

$ev: M^* \otimes M \to k;\ f \otimes m \mapsto f(m)$

$coev: k \longrightarrow M \otimes M^*;\quad 1 \mapsto \sum_i m_i \otimes m_i^*$

Write $\tau: X \otimes Y \to Y \otimes X;\ x \otimes y \longmapsto y \otimes x$

A is <u>commutative</u> if $m \circ \tau = m$, it is cocommutative if $\tau \circ \Delta = \Delta$.

If A is cocommutative then τ is an A-module map. A symmetric structure on a monoidal category is a morphism $\tau: A \otimes B \to B \otimes A$ s.t. $\tau^2 = \mathbb{1}$. So if A is cocommutative, it is a symmetric structure on A-mod. In general, τ is <u>not</u> an A-mod morphism.

"universal R matrix" "charmed element"

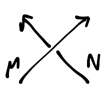
<u>Def'n</u> A ribbon Hopf algebra is a Hopf algebra with invertible elements $R \in A \otimes A$, $g \in A$ s.t.

- $\tau \circ \Delta(x) = R\, \Delta(x)\, R^{-1}$
- $(\Delta \otimes \mathbb{1})R = R_{13} R_{23}$ where $R = R_{(1)} \otimes R_{(2)}$, $R_{13} = R_{(1)} \otimes 1 \otimes R_{(2)} \in A^{\otimes 3}$
- $(\mathbb{1} \otimes \Delta)R = R_{13} R_{12}$ $\quad R_{21} = R_{(2)} \otimes R_{(1)}$
- $\Delta(g) = g \otimes g$, $S(g) = g^{-1}$, $\varepsilon(g) = 1$ "g is group-like"
  $S^2(a) = gag^{-1}$
- $m(1 \otimes g^{-1})R_{12} = m(1 \otimes g)R_{21} = v \longleftarrow$ "ribbon element"

Now we have that $\underbrace{\tau R}_{\beta_{M,N}}: M \otimes N \to N \otimes M;\ m \otimes n \mapsto R_{(2)} n \otimes R_{(1)} m$ is an A-module map.

We have:

$$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12} \qquad \text{(Quantum Yang-Baxter eq'n)}$$

Draw  for $\beta_{M,N}$. Then this satisfies Reidemeister 3.

The charmed element defines the following:

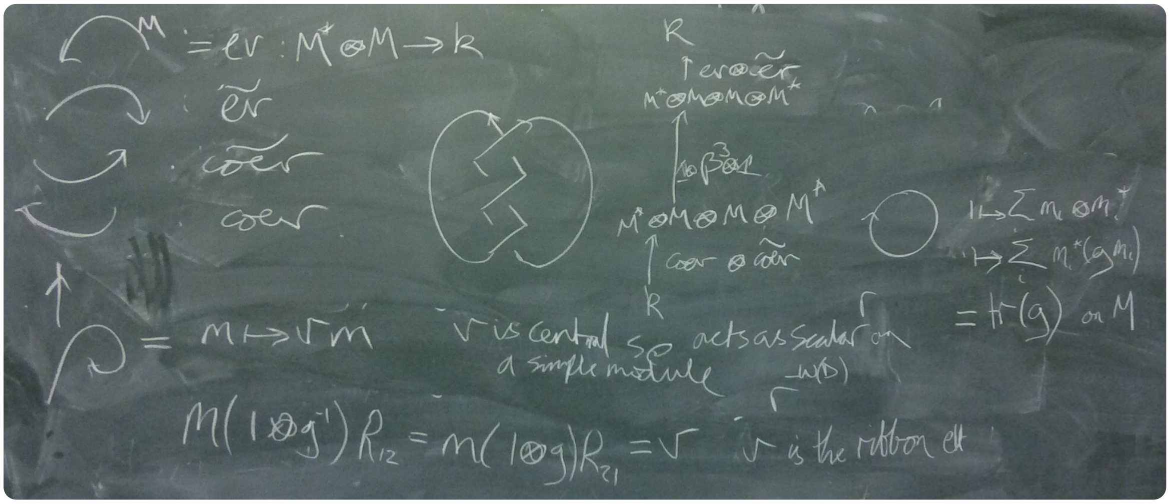
$$\widetilde{ev}: M \otimes M^* \to k;\ m_i \otimes m_i^* \longmapsto m_i^*(g m_i)$$

$$\widetilde{coev}: k \to M^* \otimes M;\ 1 \mapsto \sum m_i^* \otimes g^{-1} m_i$$

We then close up a braid and evaluate from the bottom to top.

This is the Reshetikhin–Turaev construction.

EXAMPLE



Next time we will take:

Lie algebra $\mathfrak{g} \mapsto U(\mathfrak{g}) \xmapsto{\text{quantise}} U_h(\mathfrak{g})$