Some beefs from the lecture

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- **Why doesn't the J I wrote down define a bundle map for real. manifolds?**

A: J is not defined independently of the local basis (unless the basis comes from a cx mfld):

$\boxed{\text{WRT } \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \text{ J is } J_0 = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}}$

If we change basis via $U \to GL(2n, \mathbb{R})$ we will not get a J that ~~is J² id any more~~ is $J_0 = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$ necessarily.

In our case we had transitions $U \to GL(n, \mathbb{C}) \subset GL(2n, \mathbb{R})$ so the J transforms appropriately.

- **More details on $\Omega^{p,p}_{\mathbb{R}} := \Gamma(X, \Lambda^{p,p} X \cap \Lambda^{2p} T^* X_{\mathbb{R}})$**

example $\quad -i\, dz \wedge d\bar{z} \in \Omega^{1,1} X$

but moreover $\quad -i\,(dx + i dy) \wedge (dx - i dy)$

$$= -i \cdot 2(-i\, dx \wedge dy)$$

$$= 2\, dx \wedge dy \quad \in \Lambda^{2} T^* X_{\mathbb{R}}.$$

Generally there will be forms of this type appearing in this way.

- Erratum

I gave a slightly wrong def'n of

$$\nabla^E : \Omega^r_{\mathbb{C}}(X;E) \to \Omega^{r+1}_{\mathbb{C}}(X;E)$$

for $r \neq 0$. Recall $\Omega^r_{\mathbb{C}}(X;E) = \Gamma(X, \Lambda^r T^*_{\mathbb{C}} X \otimes_{\mathbb{C}} E)$

so that $\nabla^E(\omega \otimes s) := d\omega \otimes s + (-1)^r \omega \wedge \nabla^E s$

- There was a lot of confusion about connection 1-forms

A: Conn 1-forms are in general only defined locally
Conn 2-forms are defined globally.

Given a connection

$$\nabla^E : \Omega^0_{\mathbb{C}}(X;E) \to \Omega^1_{\mathbb{C}}(X;E)$$

and a local frame $\{e_1, \dots, e_m\} \in \Gamma(U, E|_U)$, we have

$$\nabla^E e_i = \theta_i^j e_j \qquad \text{for some } \theta_i^j \in \Omega^1_{\mathbb{C}}(U)$$

Sometimes this is written as

$$\nabla^E \mathbf{e}_i = \theta \cdot \mathbf{e}_j$$

where it is understood that, $s = \sum s_i e_i$ and $\theta = [\theta_i^j]$.
we are using a matrix
of 1-forms.

Then

$$\nabla^E s = \sum \nabla^E (s_i e_i)$$

$$= \sum \left(ds_i \otimes e_i + s_i \otimes \nabla^E e_i \right)$$

$$= \sum \left(ds_j + s_i \theta_j^i \right) \otimes e_j$$

write $\nabla^E = d + \theta$ locally.

But if we change frame by $\psi : U \to GL(m, \mathbb{C})$
we get $e_i' = \psi_i^j e_j$ and $s' = \psi \cdot s$

$$\Rightarrow \theta' = \psi^{-1} d\psi \, \psi + \psi^{-1} \theta \psi \qquad \text{(by calculation in local frame).}$$

which is not the transformation of an intrinsically defined object.

<u>However</u> $\nabla^E \circ \nabla^E e_i = \Theta_i^j \otimes e_j$ for some $\Theta_i^j \in \Omega^2(U)$

and $\Theta_i^k = d\theta_i^k + \sum \theta_j^k \wedge \theta_i^j$ (again, local calculation)

<u>and</u> $\Theta' = \psi^{-1} \Theta \psi$

$\therefore$ Θ is globally well-defined $\Theta \in \Omega^2_{\mathbb{C}}(X)$.

- <u>How is this related to what Andrew did with prince bundles?</u>

If we have a splitting of TP for $G \to P \to B$

$$\begin{tikzcd} 0 \arrow[r] & \text{vert}\, TP \arrow[r] & TP \arrow[r] \arrow[l, bend right, dashed, "f"'] & \text{hor}\, TP \arrow[r] & 0 \end{tikzcd}$$

then in other words f is a section

$$f \in \Gamma(P, \mathrm{Hom}(TP, \mathrm{vert}\, TP))$$
$$= \Gamma(P, T^*P \otimes \mathrm{vert}\, TP))$$

But in this case <u>the vertical bundle is trivial</u>.

$$P \times \mathfrak{g} \xrightarrow{\cong} \mathrm{vert}\, TP; \quad (p, a) \mapsto \bar{a}(p) := \frac{d}{dt}(p \exp(tX))\Big|_{t=0}$$

$\therefore$

$$f \in \Gamma(P, T^*P \otimes (P \times \mathfrak{g}))$$

But $s \in \Gamma(P, P \times \mathfrak{g})$ is equiv just values in $\mathfrak{g}$.

$\to$ 1-form $f \in \Omega^1(P; \mathfrak{g})$

$$= \Gamma(P, T^*P \otimes (P \times \mathfrak{g}))$$

So this is a special case really.

- <u>Why are the two def'ns of holo v.b. $E \xrightarrow{\pi} X$ the same?</u>

A: In a frame $\{e_1, \dots, e_m\}$, a form $\omega \in \Omega^{p,q}(X; E)$ is

$$\omega = \sum \omega_i \otimes e_i$$

for $\omega_i \in \Omega^{p,q}(U)$. Define $\bar\partial^E \omega = \sum \bar\partial \omega_i \otimes e_i$. (*)

Conversely, the Malgrange-Koszul integrability theorem implies there are cocycles for E $(U_\alpha, \psi_{\alpha\beta})$ s.t. (*) holds.

- More details on $\Theta \in \Omega^{1,1}X$ when ∇^E is the Chern conn

Write the hermitian form in a local frame as a matrix h.
Then ∇^E unitary

$$\Rightarrow \quad dh = h\theta + \bar{\theta}^t h$$

But $(\nabla^E)^{0,1} = \bar{\partial}^E \Rightarrow \theta$ is of type $(1,0)$

$$\Rightarrow \quad h\theta = \partial h \quad \text{and} \quad \bar{\theta}^t h = \bar{\partial} h$$

$$\therefore \quad \partial\theta = \partial(h^{-1}\partial h)$$
$$= -h^{-1}\partial h \wedge h^{-1}\cdot\partial h$$
$$= -\theta\wedge\theta$$

$$\Rightarrow \quad \Theta = d\theta + \theta\wedge\theta$$
$$= \bar{\partial}\theta$$

- Erratum

$$\omega(u,v) = -\tfrac{1}{2}\,\mathrm{im}\{h(\tau(u),\tau(v))\}$$

forgot this

- ω is of type $(1,1)$

~~Facts~~ The isomorphism $\tau : TX_{\mathbb{R}} \xrightarrow{\cong} T^{1,0}X$ intertwines J and i.

$$\therefore \quad \omega(u,v) = -\tfrac{1}{2}\,\mathrm{im}\{h(\tau(u),\tau(v))\} = -\tfrac{1}{2}\,\mathrm{im}\{h(\tau(J(u)),J(v)))\}$$
$$= \omega(J(u),J(v))$$

$\Rightarrow \omega$ is type $(1,1)$.