CHARACTERISTIC CLASSES

Andrew Ranicki

Index theory seminar

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Introduction

- The **Index Theorem** identifies

$$\text{analytic index} = \text{topological index}$$

  for a differential operator on a compact manifold M which has such indices.
- The right hand side is an expression in terms of the **characteristic classes** in $H^*(M)$ of vector bundles over M, such as the tangent bundle TM.
- We shall follow Roe's book to outline the **Chern-Weil method** of constructing characteristic classes. The method uses **differential geometry** via the curvature of a connection on a vector bundle, and **algebra** via polynomial invariants of matrices.

The algebraic Chern-Weil

- There is a very nice algebraic topology construction of the Chern-Weil method in Bott's 1973 paper On the Chern-Weil homomorphism and the continuous cohomology of Lie groups in the form of a morphism from the invariant forms on the Lie algebra $\mathfrak{g}$ of a Lie group G to the cohomology of the classifying space BG

$$\phi \ : \ \mathrm{Inv}_G(S\mathfrak{g}^*) \to H^*(BG) \ .$$

- For a matrix Lie group G the Lie algebra $\mathfrak{g}$ consists of the matrices X such that $\exp(tX) \in G$ for all $t \in \mathbb{R}$.
- Let $G = U(m)$, the Lie group of unitary $m \times m$ matrices. Then $BU(m)$ is the classifying space for m-dimensional complex vector bundles. The universal Chern classes are the images

$$c_k \ = \ \phi(c_k(X)) \in H^{2k}(BU(m))$$

of the invariant polynomials $c_k(X)$ described in Lemma 2.19 below.
- However, it is the original differential geometry aspect of the 1950's Chern-Weil method which is particularly relevant to index theory

Characteristic classes

- Characteristic classes are cohomology classes used to distinguish vector bundles. Isomorphic bundles have the same characteristic classes.
- The bundles may be real or complex.
- The cohomology may be taken with various coefficients.
- A **characteristic class** γ of a vector bundle V over a space M is a collection of cohomology classes

$$\gamma_*(V) \ = \ \{\gamma_k(V) \in H^{d_k}(M) \,|\, k \geqslant 0\}$$

for some sequence $0 \leqslant d_0 < d_1 < \dots$.
- Require $\gamma_*(V)$ to be defined for every V, to satisfy:
  - (naturality) $f^*\gamma_k(V) = \gamma_k(V') \in H^{d_k}(M')$ for any bundle map $(f,b) : (M',V') \to (M,V)$,
  - (dimension) if V is m-dimensional $\gamma_k(V) = 0 \in H^{d_k}(M)$ for $d_k > m$,
- The **characteristic classes of a manifold** M are

$$\gamma_k(M) \ = \ \gamma_k(TM) \in H^{d_k}(M) \ .$$

The Euler class

- For an m-dimensional real vector bundle V over a space M the sphere and disk bundles $S(V) \subset D(V) \subset V$ define a a fibration
$$(D^m, S^{m-1}) \to (D(V), S(V)) \to M \ .$$
- An oriented V has a **Thom class**
$$U \in H^m(D(V), S(V)) \ = \ H^0(M)$$
which restricts to $1 \in H^m(D^m, S^{m-1}) = \mathbb{Z}$ on each fibre.
- The **Euler characteristic class** of V is the image
$$e(V) \ = \ [U] \in H^m(D(V); \mathbb{Z}) \ = \ H^m(M; \mathbb{Z}) \ .$$
- $e(V)$ is the primary obstruction to the existence of a non-zero section. Complete obstruction if M is an m-dimensional complex.
- The homology groups $H_*(S(V))$ of the sphere bundle $S^{m-1} \to S(V) \to M$ fit into the **Wang exact sequence**

$$\dots \to H_r(S(V)) \to H_r(M) \xrightarrow{e(V)\cap -} H_{r-m}(M) \to H_{r-1}(S(V)) \to \dots \ .$$

The prototypical index theorem

- The **Euler characteristic** of a finite-dimensional complex M is
$$\chi(M) \ = \ \sum_{r=0}^{\infty} (-)^r \dim_{\mathbb{Z}} H_r(M; \mathbb{Z}) \in \mathbb{Z} \ .$$
- **Poincaré-Hopf** The Euler class $e(M) = e(TM)$ of the tangent bundle TM of a connected oriented m-dimensional manifold M equals the Euler characteristic of M ($= 0$ if m is odd)
$$e(M) \ = \ \chi(M) \in H^m(M; \mathbb{Z}) \ = \ \mathbb{Z} \ .$$
- Can be proved by algebraic topology (e.g. Milnor-Stasheff) or by applying the Index Theorem to the operator
$$D \ = \ d + d^* \ : \ \Lambda^{odd}(T^*M) \to \Lambda^{even}(T^*M)$$
with analytic index $\chi(M)$ and topological index $e(M)$.
- A generic section of $T(M)$ has $\chi(M)$ zeros.
- TM admits a non-zero section if and only if $\chi(M) = 0$

The Stiefel-Whitney, Chern and Pontrjagin classes

- The **Stiefel-Whitney classes** of a real bundle V with $d_k = k$

$$w_k(V) \in H^k(M;\mathbb{Z}_2)\ (k \geqslant 0)\ .$$

  Depend only on the homotopy type of the sphere bundle.
- The **Chern classes** of a complex bundle V with $d_k = 2k$

$$c_k(V) \in H^{2k}(M;\mathbb{Z})\ (k \geqslant 0)\ .$$

  Index theory depends on the construction of Chern classes of complex vector bundles using connections, curvature and de Rham cohomology.
- The **Pontrjagin classes** of a real bundle V with $d_k = 4k$

$$p_k(V) \ = \ c_{2k}(V \otimes_{\mathbb{R}} \mathbb{C}) \in H^{4k}(M;\mathbb{Z})\ (k \geqslant 0)\ .$$

- Standard method: define class for $k = 0$ by convention, for $k = 1$ by construction, and extend to $k \geqslant 2$ by splitting principle and product formula. References:
  - Characteristic classes by Milnor and Stasheff.
  - Differential forms in algebraic topology by Bott and Tu.

Some examples of Chern classes

- From now on, will only consider complex vector bundles V and cohomology with complex coefficients, with Chern classes

$$c_k(V) \in H^{2k}(M) \ = \ H^{2k}(M;\mathbb{C})\ .$$

- **Example** The Chern classes of a flat (e.g. trivial) complex bundle V are

$$c_k(V) \ = \ 0 \in H^{2k}(M) \text{ for } k \geqslant 1\ .$$

  Will define flat bundles below.
- **Example** The Chern classes of a 1-dimensional (= line) complex bundle V are

$$c_k(V) \ = \ \begin{cases} e(V^{\mathbb{R}}) & \text{for } k = 1 \\ 0 & \text{for } k \geqslant 2 \end{cases} \in H^{2k}(M)$$

  with $V^{\mathbb{R}} = V$ regarded as a real 2-dimensional oriented bundle.

Some properties of the Chern classes

- **Splitting principle** For every complex m-dimensional vector bundle V over M there exists a map $f : N \to M$ such that
  - (i) $f^* : H^*(M) \to H^*(N)$ is injective.
  - (ii) $f^*V = L_1 \oplus L_2 \oplus \cdots \oplus L_m$
    is a Whitney sum of line bundles $L_1, L_2, \ldots, L_m$.
- The cup product multiplication in cohomology

$$H^i(M) \otimes H^j(M) \to H^{i+j}(M) \; ; \; x \otimes y \mapsto xy$$

  is particularly significant in the study of characteristic classes.
- **Product formula** The Chern classes of the Whitney sum of complex vector bundles V, W over M is given by

$$c_k(V \oplus W) \;=\; \sum_{i+j=k} c_i(V)c_j(W) \in H^{2k}(M) \; .$$

- Naturality, the splitting principle and the product formula show that the Chern classes of all complex vector bundles are determined by the Chern classes of line bundles.

The transition functions of a complex vector bundle

- $U(m)$ = Lie group of $m \times m$ matrices

$$A \;=\; (a_{jk} \in \mathbb{C})_{1 \leqslant j,k \leqslant m}$$

  which are **unitary**, i.e. preserve the hermitian form on $\mathbb{C}^m$:

$$AA^* = I \; , \text{ or equivalently } \sum_{\ell=1}^{m} a_{j\ell}\overline{a}_{\ell k} = \delta_{jk} \in \mathbb{C} \; .$$

- The projection of a **complex m-plane vector bundle** over M

$$\mathbb{C}^m \longrightarrow V \xrightarrow{\;p\;} M$$

  has local trivializations $(U \subset M, \phi_U : p^{-1}(U) \cong U \times \mathbb{C}^m)$ for a covering $\{U\}$ of M by open subsets $U \subseteq M$.
- For any U_1, U_2 with $U_1 \cap U_2 \neq \emptyset$ have

$$\phi_{U_2}(\phi_{U_1}^{-1}) \; : \; (U_1 \cap U_2) \times \mathbb{C}^m \to (U_1 \cap U_2) \times \mathbb{C}^m \; ; \; (x, y) \mapsto (x, h_{12}(x)(y))$$

  with $h_{12}(x) \in U(m)$ the **transition functions**.

Flat bundles I.

- **Definition** A complex vector bundle V is **flat** if the transition functions $h_{12} : U_1 \cap U_2 \to U(m)$ are locally constant.
- The higher Chern classes of a flat bundle V are trivial

$$c_k(V) \ = \ 0 \in H^{2k}(M) \text{ for } k \geqslant 1 \ .$$

- **Proposition** Let M be a connected manifold with universal cover

$$\pi_1(M) \to \widetilde{M} \to M \ .$$

A group morphism $\pi_1(M) \to U(m)$ let V determines a flat vector bundle

$$\mathbb{C}^m \to V \ = \ \mathbb{C}^m \times_{\pi_1(M)} \widetilde{M} \to M \ .$$

- **Proof** The transition functions are continuous, with

$$h_{12} \ : \ U_1 \cap U_2 \to \pi_1(M) \to U(m) \ ,$$

and $\pi_1(M)$ has the discrete topology.

Flat bundles II.

- **Proposition** A complex bundle V over a connected M is flat if and only if it is of the type

$$V \ = \ \mathbb{C}^m \times_{\pi_1(M)} \widetilde{M}$$

for some group morphism $\pi_1(M) \to U(m)$.

- **Example** Every complex bundle V over S^1 is flat, being of the form

$$\begin{aligned} V \ &= \ \mathbb{C}^m \times_{\mathbb{Z}} \mathbb{R} \\ &= \ (\mathbb{C}^m \times [0,1])/\{(z,0) \sim (u(z),1) \,|\, z \in \mathbb{C}^m\} \end{aligned}$$

for some $u \in U(m)$, with projection

$$V \to \mathbb{R}/\mathbb{Z} \ = \ S^1 \ ; \ [z,t] \mapsto e^{2\pi i t} \ .$$

The first Chern class c_1 of the tautological complex line bundle V_n over $\mathbb{CP}^n$

- The n-**dimensional complex projective space** $\mathbb{CP}^n$ is the real oriented $2n$-dimensional manifold of complex 1-dimensional subspaces $L \subset \mathbb{C}^{n+1}$, with
$$H_r(\mathbb{CP}^n) = \begin{cases} \mathbb{C} & \text{if } 0 \leqslant r \text{ even} \leqslant 2n \\ 0 & \text{otherwise} . \end{cases}$$
- The **tautological complex line bundle** over $\mathbb{CP}^n$ is
$$\mathbb{C} \to V_n = \{(L, z) \,|\, z \in L \subset \mathbb{C}^{n+1}\} \to \mathbb{CP}^n$$
with $S^1 \to S(V_n) = S^{2n+1} \to \mathbb{CP}^n$ the Hopf fibration.
- $c_1(V_n) \neq 0$ and V_n is not flat, since in view of the Wang sequence
$$\cdots \to H_r(S^{2n+1}) \to H_r(\mathbb{CP}^n) \xrightarrow{e(V_n^{\mathbb{R}}) \cap -} H_{r-2}(\mathbb{CP}^n) \to H_{r-1}(S^{2n+1}) \to \dots$$
(using $\mathbb{Z}$-coefficients) the first Chern class of V_n has to be
$$c_1(V_n) = e(V_n^{\mathbb{R}}) = 1 \in H^2(\mathbb{CP}^n) = \mathbb{C} .$$

The Chern-Weil method, following Roe

- Following the middle section of Chapter 2 of Elliptic operators, topology and asymptotic methods by John Roe, word for word.
- There are many approaches to the theory of characteristic classes. The one best suited for index theory is the so-called Chern-Weil method, using curvature to define the Chern and Pontrjagin classes with complex coefficients.
- We will consider characteristic classes with complex coefficients of complex vector bundles.
- We will represent $H^*(M) = H^*(M; \mathbb{C})$ as de Rham cohomology, that is closed forms (the kernel of d) modulo exact forms (the image of d).

Brief review of vector fields, the Lie bracket and connections

- For a manifold M let $C^\infty(M)$ be the ring of functions $f : M \to \mathbb{C}$.
- For a complex vector bundle V over M let $C^\infty(V)$ be the $C^\infty(M)$-module of sections $M \to V$.
- A **vector field** $X \in C^\infty(TM)$ is a smooth section of the tangent bundle TM, or equivalently a derivation

$$X \ : \ C^\infty(M) \to C^\infty(M) \ ; \ f \mapsto (X(f) : x \mapsto df(X(x))) \ .$$

- The **Lie bracket** $[X,Y] \in C^\infty(TM)$ of vector fields $X, Y \in C^\infty(TM)$ is

$$[X,Y] \ : \ C^\infty(M) \to C^\infty(M) \ ; \ f \mapsto X(Y(f)) - Y(X(f)) \ .$$

- A **connection** on a complex vector bundle V over M is a linear map

$$\nabla \ : \ C^\infty(TM) \otimes C^\infty(V) \to C^\infty(V) \ ; \ X \otimes Y \mapsto \nabla_X Y$$

  with $\nabla_X : C^\infty(V) \to C^\infty(V)$ the covariant derivative.
- Every complex vector bundle V admits connections ∇.

Connections, curvature and non-flatness

- In some sense, the curvature operator K of a connection ∇ on a vector bundle V measures the local deviation of V from flatness.
- If V is flat, and the base manifold M is simply-connected, then V is trivial.
- This suggests that there may be a link between curvature and characteristic classes, which measure the global deviation of V from triviality.
- Such a link is provided by the theory of invariant polynomials.

Invariant polynomials I.

- Let $\mathfrak{gl}_m(\mathbb{C})$ denote the Lie algebra of mxm matrices over $\mathbb{C}$.
- **Definition 2.18A** An **invariant polynomial** on $\mathfrak{gl}_m(\mathbb{C})$ is a polynomial function $P : \mathfrak{gl}_m(\mathbb{C}) \to \mathbb{C}$ such that

$$P(XY) \;=\; P(YX) \in \mathbb{C} \text{ for all } X, Y \in \mathfrak{gl}_m(\mathbb{C}) \;.$$

- **Example** The determinant and the trace are invariant polynomials.
- **Definition 2.18B** An **invariant formal power series** is a formal power series over $\mathfrak{gl}_m(\mathbb{C})$ each of whose homogeneous components is an invariant polynomial.
- **Lemma 2.19** The ring of invariant polynomials on $\mathfrak{gl}_m(\mathbb{C})$ is a polynomial ring generated by the polynomials

$$c_k(X) \;=\; (-2\pi i)^{-k}\mathrm{tr}(\Lambda^k X)$$

where $\Lambda^k X$ denotes the transformation induced by X on $\Lambda^k \mathbb{C}^m$

Invariant polynomials II.

- Let P be any invariant polynomial. If we first of all look at the restriction of P to diagonal matrices, we see that P must be a polynomial function of the diagonal entries. Since these diagonal entries can be interchanged by conjugation, P must in fact be a symmetric polynomial function.
- P is invariant under conjugation, so it is a symmetric polynomial function of the eigenvalues for all matrices with distinct eigenvalues: by linear algebra such matrices are conjugate to diagonal matrices.
- But the set of such matrices is dense in $\mathfrak{gl}_m(\mathbb{C})$, so a continuity argument shows that P is just a symmetric polynomial function in the eigenvalues. Now it is easy to see that $\mathrm{tr}(\Lambda^k X)$ is the kth elementary symmetric function in the eigenvalues of X.
- The main theorem on symmetric polynomials (Chapter IV of Lang's Algebra) states that the ring of symmetric polynomials is itself a polynomial ring generated by the elementary symmetric functions, and this now completes the proof.

The invariant polynomial $P(K)$ of the curvature K of a connection ∇

- The **curvature** operator K of ∇ is the 2-form on M with values in $\mathrm{End}(V)$: if $X, Y \in C^\infty(TM)$ are vector fields and $Z \in C^\infty(V)$ is a section of V, then

$$K(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z .$$

- Choosing a local framing for V, we may identify K with a matrix of ordinary 2-forms. If P is an invariant polynomial, we may apply P to this matrix to get an even-dimensional differential form $P(K)$.
- Because of the invariant nature of P, the form $P(K)$ is independent of the choice of local framing, and is therefore globally defined.

The invariant polynomial $P(K)$ as a form

- In terms of the principal $GL_m(\mathbb{C})$-bundle E associated to an m-dimensional complex vector bundle V and the terminology introduced earlier in Chapter 2, this construction may be phrased as follows.
- Let ω be the $\mathfrak{gl}_m(\mathbb{C})$ valued 1-form of the induced connection on the principal bundle E, and let Ω be the curvature form

$$(2.10) \qquad \Omega(X_1, X_2) = d\omega(X_1, X_2) + [\omega(X_1), \omega(X_2)] .$$

- Ω is a horizontal, equivariant 2-form on E with values in $\mathfrak{gl}_m(\mathbb{C})$, so $P(\Omega)$ is a horizontal invariant form on E. Such a form is the lift to E of a form on M, and this form on M is $P(K)$.

The differential form $P(K)$ I.

- Whichever approach is adopted, notice that since 2-forms are nilpotent elements in the exterior algebra, all formal power series with 2-form valued variables in fact converge. Thus, the construction makes good sense if P is merely an invariant formal power series.
- **Proposition 2.20** For any invariant polynomial (or formal power series) P, the differential form $P(K)$ is closed, and its de Rham cohomology class is independent of the choice of connection ∇ on V.
- **Proof** For the purposes of this proof let us describe an invariant formal power series P as **respectable** if the conclusion of the proposition holds for P. Clearly the sum and product of respectable formal power series are respectable. Thus, it is enough to prove that the generators defined in (2.19) are respectable. Equivalently, since

$$\det(1+qK) \;=\; \sum_k q^k \mathrm{tr}(\Lambda^k K)$$

it is enough to prove that $\det(1+qK)$, considered as a formal power series depending on the parameter q, is respectable.

The differential form $P(K)$ II.

- If P is a respectable formal power series with constant term a, and g is a function holomorphic about a, then $g \circ P$ is also a respectable formal power series. Hence, $\det(1+qK)$ is respectable if and only if $\log\det(1+qK)$ is respectable. We will now prove directly that $\log\det(1+qK)$ is respectable. For this purpose we will work in the associated principal bundle E of frames for V, with matrix-valued connection 1-form ω and corresponding curvature 2-form Ω.
- We will use the formula (2.10)

$$(**) \qquad \Omega \;=\; d\omega + \omega^2$$

where the product in the ring of matrix-valued forms is obtained by tensoring exterior product and matrix multiplication.

The differential form $P(K)$ III.

- Now suppose that ω depends on a parameter t; then Ω also depends on t, and if we use a dot to denote differentiation with respect to t, then

$$\dot{\Omega} \;=\; d\dot{\omega} + \omega\dot{\omega} + \dot{\omega}\omega \;.$$

- Consider

$$\begin{aligned} \frac{d}{dt}\log\det(1+q\Omega) &= q\,\mathrm{tr}\{(1+q\Omega)^{-1}\dot{\Omega}\} \\ &= \sum_{k=0}^{\infty}(-1)^k q^{k+1}\{\Omega^k(d\dot{\omega}+\omega\dot{\omega}+\dot{\omega}\omega)\} \end{aligned}$$

- We need the **second Bianchi identity**

$$d\Omega \;=\; \Omega\omega - \omega\Omega$$

which follows from $(**)$ on taking the exterior derivative and then substituting back for $d\omega$. We have

$$\begin{aligned} \mathrm{tr}\{\Omega^k(\omega\dot{\omega}+\dot{\omega}\omega)\} &= \mathrm{tr}\{\Omega^k\omega\dot{\omega} - \omega\Omega^k\dot{\omega}\} \text{ (by the symmetry of trace)} \\ &= \mathrm{tr}\{(d\Omega^k)\dot{\omega}\} \text{ (by the Bianchi identity)}\;. \end{aligned}$$

The differential form $P(K)$ IV.

- Therefore

$$\mathrm{tr}\{\Omega^k\omega\dot{\omega} - \omega\Omega^k\dot{\omega}\} \;=\; d\,\mathrm{tr}\{\Omega^k\dot{\omega}\}\;,$$

so

$$\frac{d}{dt}\log\det(1+q\Omega) \;=\; d\sum_{k=0}^{\infty}(-1)^k q^{k+1}\mathrm{tr}(\Omega^k\dot{\omega}\}$$

is an exact form; in fact it is the exterior derivative of a horizontal and invariant form on E. Therefore, the projection to the base manifold $(d/dt)\log\det(1+qK)$ is also exact.

- Now the result follows: for since any connection can be deformed locally to flatness, we see that $\log\det(1+qK)$ is locally exact, that is closed; and since any two connections can be linked by a (differentiable) path, the cohomology class of $\log\det(1+qK)$ is independent of the choice of connection.

The Chern classes of a complex vector bundle

- It follows from Proposition 2.20 that any invariant formal power series P defines a characteristic class for complex vector bundles. by the recipe "pick any connection and apply P to the curvature" .
- **Definition 2.21** The generators c_k defined in 2.19 correspond to characteristic classes called **Chern classes**.
- From 2.19, any characteristic class defined by an invariant polynomial is therefore a polynomial in the Chern classes.

The Pontrjagin classes of a real vector bundle

- (2.22) Suppose now that V is a *real* vector bundle, and let $V_{\mathbb{C}}$ denote its complexification,
$$V_{\mathbb{C}} \;=\; V \otimes_{\mathbb{R}} \mathbb{C} \;.$$
- The odd Chern classes of $V_{\mathbb{C}}$ are equal to zero (in complex cohomology). To see this, notice that we can give V a metric and compatible connection. The curvature F of such a connection is skew ($\mathfrak{o}(m)$-valued), so
$$\mathrm{tr}(\Lambda^k F) \;=\; (-1)^k \mathrm{tr}(\Lambda^k F) \;.$$
- The even Chern classes of $V_{\mathbb{C}}$ are called the **Pontrjagin classes** of V and are denoted
$$p_k(V) \;=\; (-1)^k c_{2k}(V_{\mathbb{C}}) \in H^{4k}(M) \;.$$
- If V is an oriented even-dimensional real vector bundle over M it also has an extra characteristic class called the **Euler class** $e(V) \in H^{\dim_{\mathbb{R}}(V)}(M)$, corresponding to the **Pfaffian** invariant polynomial. This is discussed in the exercises.

Genera I.

- Holomorphic functions can be used to build important combinations of characteristic classes. In fact, let $f(z)$ be any function holomorphic near $z = 0$. We can use f to construct an invariant formal power series Π_f, by putting

$$\Pi_f(X) = \det(\frac{-1}{2\pi i} f(X)) \ .$$

- The associated characteristic class is called the **Chern f-genus**. Notice that the Chern f-genus has the properties
  - (i) for a complex line bundle L

$$\Pi_f(L) = f(c_1(L)) \ .$$

  - (ii) for any complex vector bundles V_1 and V_2

$$\Pi_f(V_1 \oplus V_2) = \Pi_f(V_1)\Pi_f(V_2)$$

    (proved using a direct sum connection).
- In fact, it can be seen that these two properties determine the characteristic classes uniquely: this follows from the splitting principle.

Genera II.

- If the eigenvalues of the matrix $\frac{-1}{2\pi i} f(X)$ are denoted (x_j). then

$$\Pi_f(X) = \prod f(x_j)$$

  is a symmetric formal power series in the x_j, which can therefore be expressed in terms of the elementary symmetric functions of the x_j.
- But these elementary symmetric functions are just the Chern classes. Thus in the literature the genus $\Pi_f(V)$ is often written as

$$\Pi_f(V) = \prod f(x_j)$$

  where $x_1, x_2, \ldots, x_m$ are 'formal variables' subject to the relations

$$x_1 + x_2 + \cdots + x_m = c_1 \ , \ (x_1)^2 + (x_2)^2 + \cdots + (x_m)^2 = c_2 \text{ and so on} \ .$$

- In terms of the splitting principle, the formal variables x_j can be considered to represent the first Chern classes $c_1(L_j)$ of the line bundles $L_1, \ldots, L_m$ in $f^*V = L_1 \oplus \cdots \oplus L_m$.

The total Chern class

- **Example 2.23** The **total Chern class**

$$c(V) \;=\; 1 + c_1(V) + c_2(V) + \ldots$$

  is the genus associated with $f(z) = 1 + z$.
- The multiplicative law

$$c(V_1 \oplus V_2) \;=\; c(V_1)c(V_2)$$

  is the **Whitney sum** formula for the Chern classes.
- **Example 2.24** The genus associated with $f(z) = (1 + z)^{-1}I$ can be worked out by expanding the product $\prod(1 + x_j)^{-1}$ as

$$(1-x_1+(x_1)^2-\ldots)(1-x_2+(x_2)^2-\ldots)(\ldots) \;=\; 1-c_1+((c_1)^2-c_2)+\ldots$$

  If $V_1 \oplus V_2$ is trivial, this expresses the Chern classes of V_2 in terms of those of V_1

The Chern character

- (2.25) The **Chern character** ch is the characteristic class associated to the formal power series

$$X \mapsto \mathrm{tr}\exp(\frac{-1}{2\pi i}X) \; .$$

  In terms of the formal variables x_j introduced above we have

$$\mathrm{ch}(V) \;=\; \sum e^{x_j} \; .$$

- The Chern character is not a genus in the sense described above, because of the appearance of a sum rather than the product; but it does have the analogous property $\mathrm{ch}(V_1 \oplus V_2) = \mathrm{ch}(V_1) + \mathrm{ch}(V_2)$.
- Moreover, the identity $e^{x_1}e^{x_2} = e^{x_1+x_2}$ implies that

$$\mathrm{ch}(V_1 \otimes V_2) \;=\; \mathrm{ch}(V_1)\mathrm{ch}(V_2) \; .$$

- Thus, ch is a kind of "ring homomorphism". Direct calculation of the first few terms yields $\mathrm{ch}(V) = (\dim V) + c_1 + \frac{1}{2}(c_1^2 - 2c_2) + \ldots$.

The Pontrjagin character I.

- (2.26) There is an analogous theory of genera for real vector bundles. Let g be holomorphic near 0, with $g(0) = 1$. Let f be the branch of
$$z \mapsto (g(z^2))^{1/2}$$
which has $f(0) = 1$. Notice that f is an even function of z and therefore the associated genus involves only the even Chern classes.
- By definition, the **Pontrjagin g-genus** of a real vector bundle V is the Chern f-genus of its complexification $V \otimes \mathbb{C}$. The appearance of the various squares and square roots is explained by the following lemma.
- **Lemma 2.27** Let g be as above. Then for a real vector bundle V, the Pontrjagin g-genus is equal to
$$\prod_j g(y_j)$$
where the Pontrjagin classes of V are the elementary symmetric functions in the formal variables y_j

The Pontrjagin character II.

- **Proof** Regard this as an identity between invariant polynomials over $\mathfrak{o}(n)$ (= the Lie algebra of all skew-symmetric real $n \times n$ matrices). Any matrix in $\mathfrak{o}(n)$ is similar to one in block diagonal form, where the blocks are 2×2 and are of the form $X = \begin{pmatrix} 0 & \lambda \\ -\lambda & 0 \end{pmatrix}$ with eigenvalues $i\lambda$. Since both sides of the desired identity are multiplicative for direct sums, it is enough to prove it for this block X. Now
$$c_1(X) \;=\; 0\;,\; c_2(X) \;=\; \frac{1}{(2\pi i)^2}(i\lambda)(-i\lambda) \;=\; -\frac{\lambda^2}{4\pi^2}\;.$$
Thus $y = p_1(X) = \lambda^2/4\pi^2$. On the other hand, X is similar over $\mathbb{C}$ to $\begin{pmatrix} -i\lambda & 0 \\ 0 & i\lambda \end{pmatrix}$ and so
$$\Pi_f(X) \;=\; f(\frac{-\lambda}{2\pi})f(\frac{\lambda}{2\pi}) \;=\; f(\lambda/2\pi)^2 \;=\; g(\lambda^2/4\pi^2) \;=\; g(y)$$
as required

The $\widehat{\mathcal{A}}$-genus and the $\mathcal{L}$-genus

- As in the complex case, one can also interpret this lemma in terms of an appropriate splitting principle; one can take a suitable pull-back of V to split as a direct sum of real 2-plane bundles, and the y_j are then the first Pontrjagin classes of the summands.
- **Example 2.28** Two important examples are the **$\widehat{\mathcal{A}}$-genus** $\widehat{\mathcal{A}}(V)$, which is the Pontrjagin genus associated to the holomorphic function

$$z \mapsto \frac{\sqrt{z}/2}{\sinh \sqrt{z}/2} \ ,$$

and the **Hirzebruch $\mathcal{L}$-genus** $\mathcal{L}(V)$, which is the Pontrjagin genus associated with the holomorphic function

$$z \mapsto \frac{\sqrt{z}}{\tanh \sqrt{z}} \ .$$

As we will see, these combinations of characteristic classes arise naturally in the proof of the Index Theorem.