Index Theory Seminars

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1 Episode 1 - Michael Singer

This talk was given to motivate and contextualise the series. More specifically, to explain why we are interested in Dirac operators and their analytic properties.

1.1 Partial Differential Operators

Definition 1.1. A *differential operator order* m on $(\mathbb{R}, (x_1, ..., x_n))$ is a function P of the form:

$$P(x, \partial) = \sum_{|\alpha| \leq m} a_\alpha(x) \partial^\alpha$$

where α is a multi-index with $|\alpha| = \sum_i \alpha_i$ and

$$\partial^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} ... \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}.$$

We also insist the a_α are smooth.

The *symbol* σ of P at $x \in \mathbb{R}^n$ is defined by isolating the highest order derivatives:

$$\sigma(P)(x, \xi) = \sum_{|\alpha| = m} a_\alpha(x)(i\xi)^\alpha.$$

P is *elliptic* at x if $\sigma(P)(x, \xi) \neq 0$, $\forall \xi \neq 0$.

Example 1.2.

- The Laplacian $\Delta = -\sum {\partial_i}^2$ has symbol

$$\sigma(\Delta)(\xi) = \xi_1^2 + ... + \xi_n^2,$$

  and is therefore elliptic.
- The Dirac operator is elliptic (PROVE THIS).

Now suppose M is a compact manifold and $E, F \to M$ are complex vector bundles. The definitions above are local and hence extend to manifolds in the obvious way with

$$P : \Gamma(M, E) \to \Gamma(M, F).$$

From now we will consider P to be a differential operator on manifolds. Note that on manifolds we will require that σ is an invertible matrix (not just non-zero), so if P is elliptic we will certainly have $\text{rank}(E) = \text{rank}(F)$.

Definition 1.3. Equip M with a Riemannian metric. P is *Fredholm* if

$$\begin{aligned} \ker(P) &= \{u \in \Gamma(M, E) | Pu = 0\}, \\ \text{coker}(P) &= P\Gamma(M, E)^\perp \end{aligned}$$

are both finite dimensional.

Given a fibrewise inner-product $(\ , \ )$ on E we get a global L^2-inner-product on $\Gamma(M, E)$ given by

$$\langle u, u' \rangle_E = \int_M (u(x), u'(x)) d\mu_M.$$

Hence there is a formal adjoint with respect to $\langle \ , \ \rangle$

$$P^* : \Gamma(M, F) \to \Gamma(M, E)$$

such that $\langle v, Pu \rangle_F = \langle P^* v, u \rangle_E$.

Claim 1.4. P^* is elliptic if P is elliptic.

Claim 1.5. $\text{coker}(P) \cong \ker(P^*)$.

1.2 The Index Problem

Definition 1.6. The *index* of P is

$$\begin{aligned}\mathrm{Ind}(P) &= \dim\ker(P) - \dim\operatorname{coker}(P)\\ &= \dim\ker(P) - \dim\ker(P^*)\end{aligned}$$

The index of P is a rather stable quantity and depends only on the signature of P.

Example 1.7. (Finite dimensional) Consider a linear function

$$A : \mathbb{C}^m \to \mathbb{C}^n.$$

Then there is a decomposition

$$\begin{aligned}\mathbb{C}^m &= (\ker A) \bigoplus V\\ \mathbb{C}^n &= (\operatorname{im} A) \bigoplus W\end{aligned}$$

Then $A|_V$ gives an isomorphism $V \cong \operatorname{im} A$.

$$\begin{aligned}\dim\ker A &= m - \dim V\\ &= m - \dim\operatorname{im} A\\ &= m - (n - \dim W)\end{aligned}$$

$$\implies \mathrm{Ind}(A) = m - n$$

Remark 1.8. If $\mathrm{Ind}(P) > 0$ then P has at least 1 non-trivial solution.

The **Index Problem** is to calculate $\mathrm{Ind}(P)$ in topological terms.

Why would this be a sensible thing to do? Examples of quantities that can be expressed as indices include:

- The Euler characteristic $\chi(M)$ of M.
- The signature $\operatorname{sgn}(M)$ of M, when M is even dimensional.
- Before the solution of the index problem, the index of the classical Dirac operator D was known to be expressible in topological terms:
  $$\mathrm{Ind}\, D = \langle \hat{A}(M), [M]\rangle$$
  for M a spin manifold. ($\hat{A}$ is the topological quantity the '$\hat{A}$-genus'.)

So there was certainly some evidence that the index itself was a topological quantity.

1.3 The Heat-Kernel Proof

The method followed by John Roe's book is not the original proof by Atiyah and Singer. Here is an illustrative example of the method we will follow in this seminar series to prove the Index Theorem.

Example 1.9. (Finite dimensional again) We will reprove the Index Theorem for the finite dimensional case in an unnatural way.

Let $L : \mathbb{C}^{m+n} \to \mathbb{C}^{m+n}$ be given by

$$L = \begin{pmatrix} 0 & A^* \\ A & 0 \end{pmatrix}$$

so that

$$L^2 = \begin{pmatrix} AA^* & 0 \\ 0 & A^*A \end{pmatrix}$$

As $\langle u, A^*Au\rangle = 0$, we have $\|Au\|^2 = 0$. Hence

$$\begin{aligned} \ker A^*A &= \ker A, \\ \ker AA^* &= \ker A^*. \end{aligned}$$

Let $\epsilon : \mathbb{C}^{m+n} \to \mathbb{C}^{m+n}$ be given by

$$\epsilon = \begin{pmatrix} I_m & 0 \\ 0 & -I_n \end{pmatrix}$$

and define a new function $f(t) = \mathrm{tr}(\epsilon e^{-tL^2})$.

Claim 1.10.

1. f is constant for $0 \le t < \infty$.
2. $f(0) = m - n$.
3. $f(\infty) = \mathrm{Ind}(A)$.

Proof.

1. $$\begin{aligned} f'(t) &= \mathrm{tr}(\epsilon(-L^2)e^{-tL^2}) \\ &= \mathrm{tr}(-\epsilon L e^{-tL^2} L) \end{aligned}$$

   We may calculate that $\epsilon L = -L\epsilon$. But $\mathrm{tr}(\epsilon L) = \mathrm{tr}(L\epsilon)$. Hence $f'(t) = 0$.
2. $f(0) = \mathrm{tr}(\epsilon) = m - n$.
3. Let e_j be a basis of $\mathbb{C}^m$ by eigenvectors of A^*A ordered by their corresponding eigenvalues λ_j so that $\lambda_1 \le \lambda_2 \le ... \le \lambda_m$. Then

   $$A^*A = \begin{pmatrix} O_{k\times k} & & & \\ & \lambda_{k+1} & & \\ & & \ddots & \\ & & & \lambda_m \end{pmatrix}$$

   and

   $$e^{-tA^*A} = \begin{pmatrix} 1 & & & \\ & e^{-t\lambda_{k+1}} & & \\ & & \ddots & \\ & & & e^{-t\lambda_m} \end{pmatrix}.$$

   This gives us

   $$\begin{aligned} \mathrm{tr}(e^{-tAA^*}) &= \dim\ker A + O(e^{-t\lambda_{k+1}}) && \text{as } t \to \infty, \\ \mathrm{tr}(e^{-tA^*A}) &= \dim\ker A^* + O(e^{-t\lambda_{k+1}}) && \text{as } t \to \infty. \end{aligned}$$

   So

   $$\begin{aligned} f(\infty) &= \lim_{t\to\infty}(\dim\ker A - \dim\ker A^* + O(e^{-t\lambda_{k+1}})) \\ &= \mathrm{Ind}(A). \end{aligned}$$

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We want to use this argument for a Dirac operator D. In this case

$$L = \begin{pmatrix} 0 & D^* \\ D & 0 \end{pmatrix}$$

and

$$L^2 = \begin{pmatrix} D^*D & 0 \\ 0 & DD^* \end{pmatrix} = \begin{pmatrix} \nabla^*\nabla + K_1 & 0 \\ 0 & \nabla^*\nabla + K_2 \end{pmatrix}.$$

In the argument above, we used that $\frac{d}{dt}e^{-tL^2} = -L^2 e^{-tL^2}$. So we will use an analagous idea:

$$\frac{\partial}{\partial t}k(x, y; t) = -L_x^2 k(x, y; t)$$

for k the 'heat kernel' of the Dirac operator. There will also be some analysis involved to show:

- $f(t)$ is well defined.
- $f'(t) = 0$.
- $\lim_{t\to\infty} f(t) = \mathrm{Ind}(D)$.
- What happens when $t \to 0$.

For this we will need to understand:

$$\lim_{t\to\infty} \int_M k(x, y; t).$$

Remark 1.11. k has a series expansion

$$k(x, y; t) = t^{-\frac{n}{2}} e^{-\frac{|x-y|^2}{2t}} (1 + a_1(x, y)t^{\frac{1}{2}} + ...)$$

. We will be interested in the coefficient $a_{n/2}$ in the expansion. Somewhere inside this term is the $\hat{A}$-genus.