# Index Theory Seminars

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# 2 Connections and Principal Bundles - Paul Reynolds

## 2.1 Principal Bundles

The aim of this talk is to understand/recap the idea of a connection on a principal bundle and associated vector bundles.

**Definition 2.1.** Let $G$ be a Lie group, $P$ a smooth manifold

$$P \times G \to P$$

a smooth, free and proper (right) action. Then $M = P/G$ is a smooth manifold and $P \to M$ is a locally trivial submersion. We call $P$ a *principal $G$-bundle* over $M$.

**Example 2.2.**

- If $G < H$ a closed Lie subgroup then $H \to H/G$ is a principal $G$-bundle.
- $SO_3 \to SO_3/SO_2$ is a principal $SO_2$-bundle.
- A *frame* at $x \in M$ is an isomorphism $\mathbb{R}^n \to T_xM$. The bundle with fibre the collection of frames at $x$ and transition functions coming from those of the tangent bundle is denoted $GL(M)$. It is a principal $GL_n$-bundle over $M$.
- If $M$ is oriented and with a metric, we may restrict to the bundle of orthonormal frames matching that orientation. Denote it $SO(M)$ - it is a principal $SO_n$ bundle.
- The universal cover $\tilde{M} \to M$ is naturally a principal $\pi_1(M)$-bundle with the action given by the deck transformations.
- For $n \geq 3$, $\pi_1(SO_n) = \mathbb{Z}_2$, hence the universal cover $Spin_n \to SO_n$ is 2:1. A spin structure on $M$ is a principal $Spin_n$-bundle such that the following diagram commutes:

$$\begin{tikzcd}
Spin_n \arrow[r] \arrow[d, "2:1"'] & Spin(M) \arrow[r] \arrow[d] & M \\
SO_n \arrow[r] & SO(M) \arrow[ur] &
\end{tikzcd}$$

  A spin structure exists if the first and second Stiefel-Whitney classes, $w_1$ and $w_2$, vanish.

## 2.2 Associated Vector Bundles

We have seen that given the tangent bundle we can form a frame bundle which is a principal $GL_n$- or $SO_n$-bundle depending on our choices. In fact, given any vector bundle we can form some frame bundle in precisely the same way. We can pass back to vector bundles by the *associated bundle* construction. Moreover, all vector bundles can be described using this process by correct choice of $G$-bundle and linear representation of $G$.

Let $P \to M$ be a principal $G$-bundle and $\rho : G \to GL(V)$ be linear representation of $G$. Then there is a right action

$$\begin{array}{rcl} P \times V \times G & \to & P \times V \\ (p, v)g & \mapsto & (pg, \rho(g)^{-1}v). \end{array}$$

**Definition 2.3.** The associated vector bundle of $P$ by $\rho$ is the quotient

$$P \times_\rho V = (P \times V)/G.$$

Elements of $P \times_\rho V$ are written $[p, v]$ such that

$$[pg, v] = [p, \rho(g)v].$$

$P \times_\rho V$ is a vector bundle over $M$ in the following way. Given $p_x, q_x$ in the fibre above $x \in M$ there is $g \in G$ such that $p_x = q_x g$. Hence we can add fibrewise:

$$\begin{aligned} [p_x, v] + [q_x, w] &= [p_x, v] + [p_x g, w] \\ &= [p_x, v + \rho(g) w] \end{aligned}$$

**Example 2.4.**

- Set $\rho$ be the trivial representation. Then $P \times_\rho V$ is trivial over $M$.
- Set $\rho : GL_n \to GL_n$, then
$$GL(M) \times_\rho \mathbb{R}^n = TM.$$
- Set $\rho : SO_n \to GL_n$ the natural representation. Then
$$SO(M) \times_\rho \mathbb{R}^n = TM.$$
  The inner product on $\mathbb{R}^n$ in fact induces the metric on $M$.
- Set $\rho : Spin_n \to GL(\Delta_n)$ the complex spin representation with $\Delta_n$ the space of spinors. Then the complex spinor bundle is
$$\mathbb{S}M = Spin(M) \times_\rho \Delta_n.$$
- Set $\rho : Spin_n \to SO_n$, the twisted adjoint representation (equiv. double cover), then
$$Spin(M) \times_\rho \mathbb{R}^n = TM.$$

## 2.3 Connections

In order to understand and work with the smooth properties of $P$ and its associated vector bundles we will need to understand its tangent bundle $TP$ and define the idea of connection on a principal bundle. We will need to extend this in a way that is compatible with the associated bundle construction.

**Definition 2.5.** Let $P$ be a principal $G$-bundle. Then let $\mathcal{V}_p$ be the subspace of $T_pP$ generated by elements

$$\frac{d}{dt}|_{t=0} p \exp(tx)$$

where $x \in \mathfrak{g}$ (the Lie algebra of $G$). These form a bundle $\mathcal{V}$ called the *vertical bundle*.

**Remark 2.6.** Let $P \overset{\pi}{\to} M$ be the principal bundle. We could equivalently have defined $\mathcal{V} = \ker(d\pi)$. This shows that $\mathcal{V}$ is indeed a subbundle of $TP$ and is related to the tangent space of the manifold $M$.

This leads us to naturally consider the complimentary bundle $TP/\mathcal{V}$:

**Definition 2.7.** Let $\alpha$ be a $k$-form on $P$. Then $\alpha$ is *horizontal* if $\alpha(\mathcal{V}) = 0$.

Let

$$\begin{aligned} R_g : P &\to P \\ p &\mapsto pg. \end{aligned}$$

and $(R_g)_*$ the pushforward. If $\alpha$ has values in the $G$-representation $V$, then it is *equivariant* with respect to $\rho$ if

$$(R_g)^* \alpha = \rho(g)^{-1} \alpha.$$

**Theorem 2.8.** Let $\Omega^k_{\rho,hor}(P; V)$ be the $k$-forms on $P$ with values in the $G$-representation $V$ which are equivariant and horizontal. Let $\Omega^k(M; P \times_\rho V)$ be the $k$-forms on $M$ with values in the associated bundle. Then

$$\Omega^k_{\rho,hor}(P; V) \cong \Omega^k(M; P \times_\rho V)$$

as $C^\infty(M)$-modules.

**Definition 2.9.** If $\pi : P \to M$ is the principal bundles then a *principal connection* on $P$ is a splitting of the short exact sequence

$$0 \to \mathcal{V} \to TP \to \pi^* TM \to 0$$

such that the image of the splitting map in $TP$ is a $G$-invariant subbundle. Call the choice of splitting $\mathcal{H} \cong TP/\mathcal{V}$ a *horizontal* tangent space.

Equivalently, $\mathcal{H}$ is a choice of distribution on $P$ such that

$$TP = \mathcal{V} \bigoplus \mathcal{H}$$

and such that $(R_g)_* \mathcal{H}_p = \mathcal{H}_{pg}$.

**Definition 2.10.** The *connection form* of $\mathcal{H}$ is a $\mathfrak{g}$-valued 1-form $\omega$ on $P$ such that

- $\ker(\omega) = \mathcal{H}$.
- $\omega_p : T_p P \to \mathfrak{g}$ reproduces $\mathcal{V}_p \to \mathfrak{g}$.

We wish to use these horizontal spaces to define parallel transport and so do calculus on our bundles. $\mathcal{H} \subset TP$ may or may not have integrable subdistributions, but it will certainly have integral curves. If $\gamma : (-\epsilon, \epsilon) \to M$ then $\gamma$ has a unique horizontal lift $\tilde{\gamma}$ through each point in the fibre above. By this we mean:

- $\pi \circ \tilde{\gamma} = \gamma$
- $\tilde{\gamma}'(t) \in \mathcal{H}_{\tilde{\gamma}(t)}$.

Similarly if $X$ is a vector field of $M$ then it lifts uniquely to a horizontal vector field $\tilde{X}$.

**Definition 2.11.** For $\alpha \in \Omega^k(P; V)$ the *exterior covariant derivative* is

$$P_\omega^* d(\alpha)(X_1, X_2, ..., X_{k+1}) = d\alpha(P_\omega X_1, P_\omega X_2, ..., P_\omega X_{k+1})$$

where $P_\omega : TP \to \mathcal{H}$ is projection.

Note that if $\alpha$ is equivariant then $P_\omega^* d(\alpha)$ is horizontal and equivariant, so this can be thought of as a horizontal partial derivative.

**Definition 2.12.** Let $\alpha \in \Omega^0_{\rho,hor}(P; V) \cong \Omega^0(M; P \times_\rho V)$ be a section of the associated bundle (or equivalently a function from $P$ to $V$). Then the *covariant derivative* $\nabla \alpha$ is the element of $\Omega^1(M; P \times_\rho V)$ corresponding to

$$P_\omega^* d(\alpha) \in \Omega^1_{\rho,hor}(P; V)$$

.

**Proposition 2.13.** $\nabla$ is a covariant derivative in the usual sense.

**Lemma 2.14.** If $\alpha \in \Omega^1_\rho(P; V)$ then

$$P_\omega^* d(\alpha) = d\alpha + \rho_*(\omega) \wedge \alpha.$$

**Definition 2.15.** The *curvature form* of $\omega$ is

$$\Omega = P_\omega^* d(\omega).$$

using the previous lemma we derive Cartan's second structure equation:

$$\Omega = d\omega + \frac{1}{2}[\omega \wedge \omega].$$

**Proposition 2.16.** For $\Omega^k_{\rho,hor}(P; V)$,

$$(P_\omega^* d)^2(\alpha) = \rho_* \Omega \wedge \alpha.$$

Since $\Omega \in \Omega^2_{Ad,hor}(P; \mathfrak{g})$ we get a corresponding element $K \in \Omega^2(M; End(P \times_\rho V))$.

**Proposition 2.17.** $K(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}$.