

Index Theory Seminars

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3 Clifford Bundles and Dirac Operators - Patrick Orson

3.1 Motivation

Historically the motivation for this subject comes from mathematical physics. P.A.M. Dirac was interested in finding a Lorentz-invariant wave equation $D\psi = \lambda\psi$ compatible with the Klein-Gordon equation

$$\sum_{i=0}^3 \frac{\partial \psi}{\partial x_i} = \lambda \psi.$$

Causality required that D be first order in the ‘time’ variable. In essence, Dirac sought a first-order operator whose square was the Laplacian. So in n dimensions the ansatz becomes:

$$P = \sum \gamma_i \frac{\partial}{\partial x_i}$$

such that $P^2 = \Delta$, this is satisfied if and only if

$$\gamma_i^2 = -1 \text{ and } \gamma_i \gamma_j + \gamma_j \gamma_i = 0 \ \forall i \neq j.$$

This problem cannot be solved for $\gamma_i \in \mathbb{R}, \mathbb{C}$ except in the case $n = 1$, but can be solved by taking γ_i in a matrix algebra.

Example 3.1. Solutions for low dimensions

- $n = 1$: Set $\gamma_1 = i$ then $P = i \frac{\partial}{\partial x}$.
- $n = 2$: Take γ_i in the matrix algebra $M_2(\mathbb{C})$ and

$$P = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \frac{\partial}{\partial x} + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \frac{\partial}{\partial y}.$$

Note that this makes the γ_i a representation of \mathbb{C} .

- $n = 3$: Now set the γ_i to be a representation of the quaternions in $M_2(\mathbb{C})$

$$\gamma_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \gamma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \gamma_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

In this approach we have taken elements of matrix algebras instead of simply real or complex coefficients and hence changed the space on which we were acting. If before we were interested in finding a square root of the Laplacian acting on functions $f : \mathbb{R}^n \rightarrow \mathbb{R}$ or \mathbb{C} , and this must be adjusted. Generally, the unital algebra in which $\gamma_i^2 = -1$ and $\gamma_i \gamma_j + \gamma_j \gamma_i = 0 \ \forall i \neq j$ is called the Clifford algebra Cl_n and hence we are looking for vector spaces V and representation $\kappa : Cl_n \rightarrow \text{End}(V)$.

3.2 Clifford Modules

Definition 3.2. Let (V, q) be a vector space over $F = \mathbb{R}$ or \mathbb{C} with a quadratic form. Then a Clifford algebra $Cl(V, q)$ is a unital algebra with a map $\phi : V \rightarrow Cl(V, q)$, such that $\phi(v)^2 = -q(v)1$ for each $v \in V$, that is universal among such algebras. That is, if $\psi : V \rightarrow (A)$ is a function with $\psi(v)^2 = -q(v)1$ for each $v \in V$, then there is a lift to a unique algebra homomorphism $\tilde{\psi}$

$$\begin{array}{ccc} Cl(V, q) & \xrightarrow{\tilde{\psi}} & (A) \\ \uparrow \phi & \nearrow \psi & \\ V & & \end{array}$$

Proposition 3.3. $Cl(V, q)$ exists and is unique.

Proof. For existence, we may construct $Cl(V, q)$ it as follows. Take the tensor algebra $T(V)$ and quotient by the two-sided ideal $I(q)$ generated by elements of the form $v \otimes v + q(v)|v \in V$. One can check [1] that this algebra has the desired properties.

Uniqueness follows easily from the universal property. ■

Easy facts to check:

- The map ϕ is injective. Hence we will usually omit the notation and consider elements of V to be just elements inside the Clifford algebra.
- If e_1, \dots, e_n is a basis of V then $\{e_1^{k_1} \dots e_n^{k_n} | k_i = 0 \text{ or } 1\}$ is a basis of $Cl(V, q)$. Hence $\dim Cl(V, q) = 2^n$.
- As the characteristics of \mathbb{R} and \mathbb{C} are not 2, we have a polarisation identity telling us how to commute elements of the Clifford algebra.

$$\begin{aligned} vw + wv &= -2q(v, w) \\ &= -(q(v+w) - q(v) - q(w)) \end{aligned}$$

In particular it will often be convenient to pick a pseudo-orthonormal basis of V so that basis elements of V anti-commute in $Cl(V, q)$.

What do the low-dimensional (real) Clifford algebras actually *look like*? It turns out they are very familiar objects.

Example 3.4. Let q be a non-degenerate form over \mathbb{R}^n with signature (r, s) and let e_1, \dots, e_n be a pseudo-orthonormal basis. Denote the corresponding Clifford Algebra $Cl(V, q) = Cl_{r,s}$. In the special case $s = 0$, denote the algebra Cl_n .

- $n = 1$: Cl_1 is a unital algebra with basis $1, e$ such that $e^2 = -1$. This must be \mathbb{C} . The opposite form gives us $Cl_{0,1}$ and it is easily checked that this is $\mathbb{R} \oplus \mathbb{R}$.
- $n = 2$: Cl_2 has basis $1, e_1, e_2, e_1e_2$. If we form a map to the quaternions $V \rightarrow \mathbb{H}$, with $e_1 \mapsto i$ and $e_2 \mapsto j$ then the associated algebra homomorphism $Cl_2 \rightarrow \mathbb{H}$ is in fact an isomorphism (*check*).
- $n = 3$: $Cl_3 \cong \mathbb{H} \oplus \mathbb{H} \dots$

Proposition 3.5. There is a complete classification of the real Clifford algebras and it is (in some sense) modulo 8 in each argument of the signature. All Clifford algebras are isomorphic to matrix rings over \mathbb{R}, \mathbb{C} or \mathbb{H} , or sums thereof. See [1], Table II.

3.3 Dirac Operators

We will now restrict to the case where q comes from an inner product (\cdot, \cdot) , hence the signature will be $(n, 0)$ and we will drop reference to the dimension where this causes no confusion. Moreover it will be convenient to consider complexified Clifford algebras $Cl(V) \otimes_{\mathbb{R}} \mathbb{C}$.

Definition 3.6. A *Clifford module* S for a real inner product space V to be a vector space which is a left module over $Cl(V) \otimes \mathbb{C}$. In other words, we have a *Clifford multiplication* on elements of S by elements in the Clifford algebra, that returns another element of S . I will denote the Clifford multiplication by \cdot . (One could also think of this as a representation $c : Cl(V) \otimes \mathbb{C} \rightarrow \text{End}(S)$.)

The Clifford multiplication can also be performed on elements $s \in C^\infty(V, S)$, the S -valued functions on V .

Definition 3.7. The *Dirac operator* of the Clifford module is:

$$Ds = \sum_i e_i \cdot (\partial_i s).$$

This has the property we have been trying to develop:

$$D^2 s = \sum_{i,j} e_j \cdot \partial_j (e_i \cdot \partial_i s) = \sum_{i,j} e_j \cdot (e_i \cdot (\partial_j \partial_i s)) = - \sum_i \partial_i^2 s.$$

We now wish to transfer this construction to vector bundles on a Riemannian manifold (M, g) . Each tangent space $T_m M$ has an inner product and so is a natural choice of V for the construction above.

Definition 3.8. Let S be a smooth vector bundle over M such that the fibres S_m are Clifford modules over $Cl(T_m M) \otimes \mathbb{C}$. S is a *Clifford bundle* if it is equipped with a Hermitian metric h and compatible connection ∇^S satisfying the following properties:

1. Skew adjointness: $h(v \cdot s_1, s_2) + h(s_1, v \cdot s_2) = 0$ for each $v \in TM$ and $s_1, s_2 \in S$.

2. Compatibility with Levi-Cevita:

$$\nabla_X^S (Y \cdot s) = (\nabla_X Y) \cdot s + Y \cdot \nabla_X^S s$$

for any vector fields X, Y and $s \in S$.

Remark 3.9. The condition (1) will ensure that the Dirac operator is self adjoint. The condition (2) is a natural thing to ask for in the sense that if we were to construct both TM and S as associated bundles of same principal bundle P , the connections are both push-forwards of the same connection on P .

Definition 3.10. The *Dirac operator* D of a Clifford bundle is the first order differential operator on $\Omega^0(S)$ defined by the composition

$$\Omega^0(S) \xrightarrow{\nabla^S} \Omega^1(S) \xrightarrow{\text{metric}} \Gamma(TM \otimes S) \xrightarrow{\text{Clifford}} \Omega^0(S)$$

So if we fix a local orthonormal frame e_1, \dots, e_n for TM , we get

$$s \mapsto \sum e^i \otimes \nabla_i^S s \mapsto \sum e_i \otimes \nabla_i^S s \mapsto \sum e_i \cdot \nabla_i^S s$$

i.e. locally

$$Ds = \sum e_i \cdot \nabla_i^S s.$$

Remark 3.11. For brevity, and where it causes no confusion, I will usually drop reference to which metric and connection I am using.

Proposition 3.12. (Weitzenbock Formula)

$$D^2 s = \nabla^* \nabla s + K s,$$

where ∇^* is the formal adjoint of ∇ , and K is the ‘Clifford contraction’ of the curvature 2-form on S given by $K = \sum_{i < j} c(e_i) c(e_j) K(e_i, e_j)$.

Definition 3.13. A frame over a vector bundle $V \rightarrow M$ near a point $m \in M$ is called *synchronous* at m with respect to local co-ordinates near m if all the connection coefficients vanish at m . Such a local frame can always be chosen by taking a frame above m with this property and parallel transporting it along radial lines.

Pick a local frame e_1, \dots, e_n of TM that is synchronous at $m \in M$ (with respect to local co-ordinates) and let $s \in C^\infty(S)$. Then, as the connection coefficients vanish, $\nabla_i e_j = 0$ and $[e_i, e_j] = 0$ for all i, j . Then at m :

$$\begin{aligned} D^2 s &= \sum_{i,j} e_i \cdot \nabla_i (e_j \cdot \nabla_j s) \\ &= \sum_{i,j} e_i \cdot e_j \cdot \nabla_i \nabla_j s \\ &= - \sum_i \nabla_i^2 s + \sum_{i < j} e_i \cdot e_j \cdot (\nabla_i \nabla_j - \nabla_j \nabla_i) s \end{aligned}$$

The first term is Laplacian-like and by a calculation in local co-ordinates it can be verified that it is indeed $\nabla^* \nabla$ (see [2]). The curvature 2-form is given in local co-ordinates by $K(e_i, e_j) = \nabla_i \nabla_j - \nabla_j \nabla_i - \nabla_{[\partial_i, \partial_j]}$. So at m the second term in the equation above is the curvature.

Proposition 3.14. D is self adjoint.

3.4 Examples of Clifford Bundles

3.4.1 Regular Representation

Fix a vector space V with an inner product. $Cl(V) \cong \bigwedge^* V$ as vector spaces (but not as algebras). Further, $\bigwedge V \cong \bigwedge^* V$ so there is an isomorphism $\phi : Cl(V) \rightarrow \bigwedge^* V$. We can exploit this fact to build a Clifford module. $Cl(TM) \otimes \mathbb{C}$ acts on itself on the left by multiplication. Hence $S = \bigwedge^* V \otimes \mathbb{C}$ is a Clifford module with representation:

$$c : \bigwedge^* TM \rightarrow \text{End}_{\mathbb{C}}(\bigwedge^* TM)$$

given by $c(v)w = \phi(v \cdot w)$. We can express this more concretely in terms of the interior and exterior products in $\bigwedge^* TM$ (see [3] for details of the interior product ι).

Claim 3.15. Let $e, \omega \in \Omega^k(M)$, then $c(e)\omega = e \wedge \omega - \iota(e)\omega$.

Proof. Calculation. ■

To check that this forms a Clifford bundle we still need the compatibility conditions for the metric and connection. These are shown in [2]. What is the associated Dirac operator?

$$\begin{aligned} D\omega &= \sum_i c(e_i) \nabla_i \omega \\ &= \sum_i e_i \wedge \nabla_i \omega + \iota(e_i) \nabla_i \omega \\ &= d\omega + d^* \omega \end{aligned}$$

(the de-Rham operator) and

$$D^2 = dd^* + d^*d$$

3.4.2 Spin Representation

Suppose V is a vector space with $\dim_{\mathbb{R}} V = 2m$ and $J : V \rightarrow V$ is a complex structure. On top of this, complexify V . Then we may always decompose

$$V \otimes_{\mathbb{R}} \mathbb{C} = P \oplus Q$$

as the $\pm i$ eigenspaces of J . P and Q are the maximal *isotropic* subspaces (i.e. $(p_1, p_2) = (q_1, q_2) = 0$ for $p_i \in P$ and $q_i \in Q$).

Our intended Clifford module is $\bigwedge^* P$ and our Clifford algebra is $Cl(V) \otimes \mathbb{C}$. Let $p + q \in V \otimes \mathbb{C}$, then action can be defined as:

$$(p + q) \cdot x = \sqrt{2}(p \wedge x - \iota(q)x).$$

This extends to an action of $Cl(V) \otimes \mathbb{C}$.

When can we extend this process to bundles? We would require at least a $2m$ -dimensional Riemannian manifold with an almost complex structure $J : TM \rightarrow TM$ and compatible metric. One can check that if the manifold M is in fact complex itself, then with a compatible Hermitian metric and appropriate connection, $\bigwedge^* T_{\mathbb{C}}M$ (the bundle form of P) is a Clifford bundle.

Theorem 3.16. If M is Kähler then $D = \sqrt{2}(\bar{\partial} + \bar{\partial}^*)$.