# Index Theory Seminars

February 22, 2011

# 5 Spectral Properties of the Dirac Operator - Mark Powell

## 5.1 Outline

Let $S$ be a Clifford bundle on $M$ compact. Let $D$ be such that $D^2 = \nabla^*\nabla + B$ where $B$ is some first order operator. **We wish to show that $e^{-tD^2}$ makes sense as an operator.**

$D$ is an operator on $C^\infty(S)$ but we want to work with the properties of a Hilbert space.

IDEA: We pass back and forth between $C^\infty(S)$ and $L^2(S)$.

$$\begin{tikzcd}
D : C^\infty(S) \arrow[r] \arrow[d] & C^\infty(S) \arrow[d] \\
\bar{D} : L^2(S) \arrow[r] & L^2(S)
\end{tikzcd}$$

We will use nice properties of compact operators on Hilbert spaces to get a spectral decomposition and show that what we got was smooth after all.

**Theorem 5.1.** (Main Theorem of talk) There is an orthogonal decomposition

$$L^2(S) = \bigoplus_{\lambda \in \Lambda} L^2(S)_\lambda$$

into countably many finite-dimensional eigenspaces of smooth sections of $D$. Moreover $\Lambda \subseteq \mathbb{R}$ is discrete.

Consequently (using functional calculus), for $s \in L^2(S)$

$$s = \sum_{\lambda \in \Lambda} s_\lambda,$$

then we may *define*

$$e^{-tD^2}(s) = \sum_{\lambda \in \Lambda} e^{-t\lambda^2} s_\lambda.$$

## 5.2 Tools for the Proof

**Definition 5.2.** (Rough) The *Sobolev space* $W^k$ is the space of sections of $S$ whose first $k$ derivatives are in $L^2(S)$.

$$C^\infty(S) \subseteq ... \subseteq W^k \subseteq ... \subseteq W^1 \subseteq W^0 = L^2(S)$$

More precisely $W^k$ is the completion of $C^\infty(S)$ is $|| \cdot ||_k$ (the Sobolev $k$-norm).

$$||f||_k = \sum_{|\alpha| \leq k} ||\frac{\partial f}{\partial x^\alpha}||_{L^2}$$

Moreover, $\bigcap_k W^k = C^\infty(S)$ by the *Sobolev Embedding Theorem.*

**Lemma 5.3.** (Garding's Inequality) **5.14** [1]

$$||s||_1 \leq C(||s||_0 + ||Ds||_0)$$

for some $C > 0$. This is to say

$$\int (|s|^2 + |\nabla s|^2) \leq C(\int |s|^2 + \int |Ds|^2).$$

A generalisation of this is **5.16** [1]

$$||s||_{k+1} \leq C_k(||s||_k + ||Ds||_k).$$

So in particular, if $s \in W^k$ and $Ds = 0$ then $s \in W^{k+1}$. By induction this will allow us to pass to $s \in C^\infty(S)$.

**Proposition 5.4.** (**5.24** [1]) $\ker(\bar{D})$ comprises smooth sections.

**Theorem 5.5.** (Rellich) The inclusion $W^{k+1} \to W^k$ is a compact operator.

## 5.3 Proof of Main Theorem

**Definition 5.6.** Let $G$ be the graph of $D$

$$G = \{(x, Dx) \in L^2(S) \oplus L^2(S) | x \in C^\infty(S)\}$$

and let $\bar{G}$ be the closure of $G$. $\bar{G}$ is also a graph (by Closed Graph Theorem). Define the operator $\bar{D}$ by this - it has domain $W^1$ due to Garding's Inequality.

$$\bar{D} : W^1 \to L^2(S)$$

**Definition 5.7.** Define $Q : L^2(S) \to W^1$ in the following way. Let $x \in L^2(S)$, then let $Qx$ be such that $(Qx - x, \bar{D}(Qx)) \perp \bar{G}$, the perpendicular projection of $(x, 0)$ to $\bar{G}$.

- As $Q$ is an orthogonal projection, it is self-adjoint.
- $||x||^2 = ||Qx||^2 + ||\bar{D}Qx||^2$, so Garding's Inequality implies that $Q$ is bounded.
- Recalling Rellich's Theorem, we see that

  $$G : L^2(S) \to W^1 \to L^2(S)$$

  is compact and self adjoint.
- Spectral Theorem gives us

  $$L^2(S) = \bigoplus_\rho L^2(S)_\rho,$$

  a decomposition by eigenspaces of $Q$ with discrete eigenvalues $\rho$, where $\rho \to 0$.

**Claim 5.8.** $G^\perp = JG$ for $J : (x, y) \mapsto (y, -x)$.

*Proof.* Let $(x, y) \in G^\perp$, then

$$\begin{array}{rrcll} & \langle (x,y), (s, Ds) \rangle & = & 0 & \forall s \in C^\infty(S) \\ \implies & \langle x, s\rangle + \langle y, Ds\rangle & = & 0 & \\ \implies & \langle x + Dy, s\rangle & = & 0 & \text{as } D = D^* \\ & & & & \\ \implies & \bar{D}y = -x & & & \end{array}$$

■

We may now proceed to prove the Main Theorem.

*Proof.*

$$\begin{array}{rcl} L^2(S) \oplus L^2(S) & = & \bar{G} \oplus \bar{G}^\perp \\ & = & \bar{G} \oplus G^\perp \\ & = & \bar{G} \oplus JG \\ & = & \bar{G} \oplus J\bar{G} \end{array}$$

Let $x \in L^2(S)$ be an eigenvector of $Q$, then there exists $y \in L^2(S)$ such that

$$\begin{array}{rcl} (x, 0) & = & (Qx, \bar{D}Qx) + (-\bar{D}y, y) \\ & = & \rho^2(x, \bar{D}x) + (\bar{D}y, y) \end{array}$$

So $(\rho^2 - 1)x = \bar{D}y$ and $y = -\rho^2\bar{D}x$. Let $\lambda^2 = (1 - \rho^2)/\rho^2$ and $z = -(1/\rho^2\lambda)y$. We may calculate that

$$\begin{array}{rcl} \bar{D}x & = & \lambda z \\ \bar{D}z & = & \lambda x \end{array}$$

So $x \pm z$ are eigenvectors of $\bar{D}$ with eigenvalues $\pm\lambda$.

Hence $s_\lambda \in \ker(D - \lambda)$ and therefore $s_\lambda \in C^\infty(S)$, ■

# References

[1] John Roe. *Elliptic operators, topology and asymptotic methods*, volume 395 of *Pitman Research Notes in Mathematics Series*. Longman, Harlow, second edition, 1998.