Index Theory Seminars

March 1, 2012

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1 Episode 1 - Michael Singer

This talk was given to motivate and contextualise the series. More specifically, to explain why we are interested in Dirac operators and their analytic properties.

1.1 Partial Differential Operators

Definition 1.1. A *differential operator order* m on $(\mathbb{R}, (x_1, ..., x_n))$ is a function P of the form:

$$P(x, \partial) = \sum_{|\alpha| \leq m} a_\alpha(x) \partial^\alpha$$

where α is a multi-index with $|\alpha| = \sum_i \alpha_i$ and

$$\partial^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} ... \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}.$$

We also insist the a_α are smooth.

The *symbol* σ of P at $x \in \mathbb{R}^n$ is defined by isolating the highest order derivatives:

$$\sigma(P)(x, \xi) = \sum_{|\alpha| = m} a_\alpha(x)(i\xi)^\alpha.$$

P is *elliptic* at x if $\sigma(P)(x, \xi) \neq 0$, $\forall \xi \neq 0$.

Example 1.2.

- The Laplacian $\Delta = -\sum {\partial_i}^2$ has symbol

$$\sigma(\Delta)(\xi) = \xi_1^2 + ... + \xi_n^2,$$

  and is therefore elliptic.

- The Dirac operator is elliptic (PROVE THIS).

Now suppose M is a compact manifold and $E, F \to M$ are complex vector bundles. The definitions above are local and hence extend to manifolds in the obvious way with

$$P : \Gamma(M, E) \to \Gamma(M, F).$$

From now we will consider P to be a differential operator on manifolds. Note that on manifolds we will require that σ is an invertible matrix (not just non-zero), so if P is elliptic we will certainly have $\text{rank}(E) = \text{rank}(F)$.

Definition 1.3. Equip M with a Riemannian metric. P is *Fredholm* if

$$\begin{aligned} \ker(P) &= \{u \in \Gamma(M, E) | Pu = 0\}, \\ \text{coker}(P) &= P\Gamma(M, E)^\perp \end{aligned}$$

are both finite dimensional.

Given a fibrewise inner-product $(\ ,\ )$ on E we get a global L^2-inner-product on $\Gamma(M, E)$ given by

$$\langle u, u' \rangle_E = \int_M (u(x), u'(x)) d\mu_M.$$

Hence there is a formal adjoint with respect to $\langle\ ,\ \rangle$

$$P^* : \Gamma(M, F) \to \Gamma(M, E)$$

such that $\langle v, Pu \rangle_F = \langle P^* v, u \rangle_E$.

Claim 1.4. P^* is elliptic if P is elliptic.

Claim 1.5. $\text{coker}(P) \cong \ker(P^*)$.

1.2 The Index Problem

Definition 1.6. The *index* of P is

$$\begin{aligned} \mathrm{Ind}(P) &= \dim\ker(P) - \dim\mathrm{coker}(P) \\ &= \dim\ker(P) - \dim\ker(P^*) \end{aligned}$$

The index of P is a rather stable quantity and depends only on the signature of P.

Example 1.7. (Finite dimensional) Consider a linear function

$$A : \mathbb{C}^m \to \mathbb{C}^n.$$

Then there is a decomposition

$$\begin{aligned} \mathbb{C}^m &= (\ker A) \bigoplus V \\ \mathbb{C}^n &= (\mathrm{im}\, A) \bigoplus W \end{aligned}$$

Then $A|_V$ gives an isomorphism $V \cong \mathrm{im}\, A$.

$$\begin{aligned} \dim\ker A &= m - \dim V \\ &= m - \dim\mathrm{im}\, A \\ &= m - (n - \dim W) \end{aligned}$$

$$\Longrightarrow \mathrm{Ind}(A) = m - n$$

Remark 1.8. If $\mathrm{Ind}(P) > 0$ then P has at least 1 non-trivial solution.

The **Index Problem** is to calculate $\mathrm{Ind}(P)$ in topological terms.

Why would this be a sensible thing to do? Examples of quantities that can be expressed as indices include:

- The Euler characteristic $\chi(M)$ of M.
- The signature $\mathrm{sgn}(M)$ of M, when M is even dimensional.
- Before the solution of the index problem, the index of the classical Dirac operator D was known to be expressible in topological terms:
$$\mathrm{Ind}\, D = \langle \hat{A}(M), [M] \rangle$$
for M a spin manifold. ($\hat{A}$ is the topological quantity the '$\hat{A}$-genus'.)

So there was certainly some evidence that the index itself was a topological quantity.

1.3 The Heat-Kernel Proof

The method followed by John Roe's book is not the original proof by Atiyah and Singer. Here is an illustrative example of the method we will follow in this seminar series to prove the Index Theorem.

Example 1.9. (Finite dimensional again) We will reprove the Index Theorem for the finite dimensional case in an unnatural way.

Let $L : \mathbb{C}^{m+n} \to \mathbb{C}^{m+n}$ be given by

$$L = \begin{pmatrix} 0 & A^* \\ A & 0 \end{pmatrix}$$

so that

$$L^2 = \begin{pmatrix} AA^* & 0 \\ 0 & A^*A \end{pmatrix}$$

As $\langle u, A^*Au \rangle = 0$, we have $\|Au\|^2 = 0$. Hence

$$\begin{aligned} \ker A^*A &= \ker A, \\ \ker AA^* &= \ker A^*. \end{aligned}$$

Let $\epsilon : \mathbb{C}^{m+n} \to \mathbb{C}^{m+n}$ be given by

$$\epsilon = \begin{pmatrix} I_m & 0 \\ 0 & -I_n \end{pmatrix}$$

and define a new function $f(t) = \mathrm{tr}(\epsilon e^{-tL^2})$.

Claim 1.10. 1. f is constant for $0 \leq t < \infty$.

2. $f(0) = m - n$.
3. $f(\infty) = \mathrm{Ind}(A)$.

Proof. 1.

$$\begin{aligned} f'(t) &= \mathrm{tr}(\epsilon(-L^2)e^{-tL^2}) \\ &= \mathrm{tr}(-\epsilon L e^{-tL^2} L) \end{aligned}$$

We may calculate that $\epsilon L = -L\epsilon$. But $\mathrm{tr}(\epsilon L) = \mathrm{tr}(L\epsilon)$. Hence $f'(t) = 0$.

2. $f(0) = \mathrm{tr}(\epsilon) = m - n$.
3. Let e_j be a basis of $\mathbb{C}^m$ by eigenvectors of A^*A ordered by their corresponding eigenvalues λ_j so that $\lambda_1 \leq \lambda_2 \leq ... \leq \lambda_m$. Then

$$A^*A = \begin{pmatrix} O_{k\times k} & & & \\ & \lambda_{k+1} & & \\ & & \ddots & \\ & & & \lambda_m \end{pmatrix}$$

and

$$e^{-tA^*A} = \begin{pmatrix} 1 & & & \\ & e^{-t\lambda_{k+1}} & & \\ & & \ddots & \\ & & & e^{-t\lambda_m} \end{pmatrix}.$$

This gives us

$$\begin{aligned} \mathrm{tr}(e^{-tAA^*}) &= \dim\ker A + O(e^{-t\lambda_{k+1}}) && \text{as } t \to \infty, \\ \mathrm{tr}(e^{-tA^*A}) &= \dim\ker A^* + O(e^{-t\lambda_{k+1}}) && \text{as } t \to \infty. \end{aligned}$$

So

$$\begin{aligned} f(\infty) &= \lim_{t\to\infty} (\dim\ker A - \dim\ker A^* + O(e^{-t\lambda_{k+1}})) \\ &= \mathrm{Ind}(A). \end{aligned}$$

■

We want to use this argument for a Dirac operator D. In this case

$$L = \begin{pmatrix} 0 & D^* \\ D & 0 \end{pmatrix}$$

and

$$L^2 = \begin{pmatrix} D^*D & 0 \\ 0 & DD^* \end{pmatrix} = \begin{pmatrix} \nabla^*\nabla + K_1 & 0 \\ 0 & \nabla^*\nabla + K_2 \end{pmatrix}.$$

In the argument above, we used that $\frac{d}{dt}e^{-tL^2} = -L^2 e^{-tL^2}$. So we will use an analagous idea:

$$\frac{\partial}{\partial t}k(x,y;t) = -L_x^2 k(x,y;t)$$

for k the 'heat kernel' of the Dirac operator. There will also be some analysis involved to show:

- $f(t)$ is well defined.
- $f'(t) = 0$.
- $\lim_{t\to\infty} f(t) = \mathrm{Ind}(D)$.
- What happens when $t \to 0$.

For this we will need to understand:

$$\lim_{t\to\infty} \int_M k(x,y;t).$$

Remark 1.11. k has a series expansion

$$k(x,y;t) = t^{-\frac{n}{2}} e^{-\frac{|x-y|^2}{2t}} (1 + a_1(x,y)t^{\frac{1}{2}} + ...)$$

. We will be interested in the coefficient $a_{n/2}$ in the expansion. Somewhere inside this term is the $\hat{A}$-genus.

2 Connections and Principal Bundles - Paul Reynolds

2.1 Principal Bundles

The aim of this talk is to understand/recap the idea of a connection on a principal bundle and associated vector bundles.

Definition 2.1. Let G be a Lie group, P a smooth manifold

$$P \times G \to P$$

a smooth, free and proper (right) action. Then $M = P/G$ is a smooth manifold and $P \to M$ is a locally trivial submersion. We call P a *principal G-bundle* over M.

Example 2.2.

- If $G < H$ a closed Lie subgroup then $H \to H/G$ is a principal G-bundle.
- $SO_3 \to SO_3/SO_2$ is a principal SO_2-bundle.
- A *frame* at $x \in M$ is an isomorphism $\mathbb{R}^n \to T_xM$. The bundle with fibre the collection of frames at x and transition functions coming from those of the tangent bundle is denoted $GL(M)$. It is a principal GL_n-bundle over M.
- If M is oriented and with a metric, we may restrict to the bundle of orthonormal frames matching that orientation. Denote it $SO(M)$ - it is a principal SO_n bundle.
- The universal cover $\tilde{M} \to M$ is naturally a principal $\pi_1(M)$-bundle with the action given by the deck transformations.
- For $n \geq 3$, $\pi_1(SO_n) = \mathbb{Z}_2$, hence the universal cover $Spin_n \to SO_n$ is 2:1. A spin structure on M is a principal $Spin_n$-bundle such that the following diagram commutes:

$$\begin{tikzcd}
Spin_n \arrow[r] \arrow[d, "2:1"'] & Spin(M) \arrow[r] \arrow[d] & M \\
SO_n \arrow[r] & SO(M) \arrow[ur] &
\end{tikzcd}$$

A spin structure exists if the first and second Stiefel-Whitney classes, w_1 and w_2, vanish.

2.2 Associated Vector Bundles

We have seen that given the tangent bundle we can form a frame bundle which is a principal GL_n- or SO_n-bundle depending on our choices. In fact, given any vector bundle we can form some frame bundle in precisely the same way. We can pass back to vector bundles by the *associated bundle* construction. Moreover, all vector bundles can be described using this process by correct choice of G-bundle and linear representation of G.

Let $P \to M$ be a principal G-bundle and $\rho : G \to GL(V)$ be linear representation of G. Then there is a right action

$$\begin{aligned} P \times V \times G &\to P \times V \\ (p, v)g &\mapsto (pg, \rho(g)^{-1}v). \end{aligned}$$

Definition 2.3. The associated vector bundle of P by ρ is the quotient

$$P \times_\rho V = (P \times V)/G.$$

Elements of $P \times_\rho V$ are written $[p, v]$ such that

$$[pg, v] = [p, \rho(g)v].$$

$P \times_\rho V$ is a vector bundle over M in the following way. Given p_x, q_x in the fibre above $x \in M$ there is $g \in G$ such that $p_x = q_x g$. Hence we can add fibrewise:

$$\begin{aligned} [p_x, v] + [q_x, w] &= [p_x, v] + [p_x g, w] \\ &= [p_x, v + \rho(g)w] \end{aligned}$$

Example 2.4.

- Set ρ be the trivial representation. Then $P \times_\rho V$ is trivial over M.
- Set $\rho : GL_n \to GL_n$, then
$$GL(M) \times_\rho \mathbb{R}^n = TM.$$
- Set $\rho : SO_n \to GL_n$ the natural representation. Then
$$SO(M) \times_\rho \mathbb{R}^n = TM.$$
The inner product on $\mathbb{R}^n$ in fact induces the metric on M.
- Set $\rho : Spin_n \to GL(\Delta_n)$ the complex spin representation with Δ_n the space of spinors. Then the complex spinor bundle is
$$\mathbb{S}M = Spin(M) \times_\rho \Delta_n.$$
- Set $\rho : Spin_n \to SO_n$, the twisted adjoint representation (equiv. double cover), then
$$Spin(M) \times_\rho \mathbb{R}^n = TM.$$

2.3 Connections

In order to understand and work with the smooth properties of P and its associated vector bundles we will need to understand its tangent bundle TP and define the idea of connection on a principal bundle. We will need to extend this in a way that is compatible with the associated bundle construction.

Definition 2.5. Let P be a principal G-bundle. Then let $\mathcal{V}_p$ be the subspace of T_pP generated by elements

$$\frac{d}{dt}|_{t=0} p \exp(tx)$$

where $x \in \mathfrak{g}$ (the Lie algebra of G). These form a bundle $\mathcal{V}$ called the *vertical bundle*.

Remark 2.6. Let $P \xrightarrow{\pi} M$ be the principal bundle. We could equivalently have defined $\mathcal{V} = \ker(d\pi)$. This shows that $\mathcal{V}$ is indeed a subbundle of TP and is related to the tangent space of the manifold M.

This leads us to naturally consider the complimentary bundle $TP/\mathcal{V}$:

Definition 2.7. Let α be a k-form on P. Then α is *horizontal* if $\alpha(\mathcal{V}) = 0$.

Let

$$\begin{aligned} R_g : P &\to P \\ p &\mapsto pg. \end{aligned}$$

and $(R_g)_*$ the pushforward. If α has values in the G-representation V, then it is *equivariant* with respect to ρ if

$$(R_g)^*\alpha = \rho(g)^{-1}\alpha.$$

Theorem 2.8. Let $\Omega^k_{\rho,hor}(P;V)$ be the k-forms on P with values in the G-representation V which are equivariant and horizontal. Let $\Omega^k(M; P \times_\rho V)$ be the k-forms on M with values in the associated bundle. Then

$$\Omega^k_{\rho,hor}(P;V) \cong \Omega^k(M; P \times_\rho V)$$

as $C^\infty(M)$-modules.

Definition 2.9. If $\pi : P \to M$ is the principal bundles then a *principal connection* on P is a splitting of the short exact sequence

$$0 \to \mathcal{V} \to TP \to \pi^* TM \to 0$$

such that the image of the splitting map in TP is a G-invariant subbundle. Call the choice of splitting $\mathcal{H} \cong TP/\mathcal{V}$ a *horizontal* tangent space.

Equivalently, $\mathcal{H}$ is a choice of distribution on P such that

$$TP = \mathcal{V} \bigoplus \mathcal{H}$$

and such that $(R_g)_* \mathcal{H}_p = \mathcal{H}_{pg}$.

Definition 2.10. The *connection form* of $\mathcal{H}$ is a $\mathfrak{g}$-valued 1-form ω on P such that

- $\ker(\omega) = \mathcal{H}$.
- $\omega_p : T_p P \to \mathfrak{g}$ reproduces $\mathcal{V}_p \to \mathfrak{g}$.

We wish to use these horizontal spaces to define parallel transport and so do calculus on our bundles. $\mathcal{H} \subset TP$ may or may not have integrable subdistributions, but it will certainly have integral curves. If $\gamma : (-\epsilon, \epsilon) \to M$ then γ has a unique horizontal lift $\tilde{\gamma}$ through each point in the fibre above. By this we mean:

- $\pi \circ \tilde{\gamma} = \gamma$
- $\tilde{\gamma}'(t) \in \mathcal{H}_{\tilde{\gamma}(t)}$.

Similarly if X is a vector field of M then it lifts uniquely to a horizontal vector field $\tilde{X}$.

Definition 2.11. For $\alpha \in \Omega^k(P; V)$ the *exterior covariant derivative* is

$$P_\omega^* d(\alpha)(X_1, X_2, ..., X_{k+1}) = d\alpha(P_\omega X_1, P_\omega X_2, ..., P_\omega X_{k+1})$$

where $P_\omega : TP \to \mathcal{H}$ is projection.

Note that if α is equivariant then $P_\omega^* d(\alpha)$ is horizontal and equivariant, so this can be thought of as a horizontal partial derivative.

Definition 2.12. Let $\alpha \in \Omega^0_{\rho,hor}(P; V) \cong \Omega^0(M; P \times_\rho V)$ be a section of the associated bundle (or equivalently a function from P to V). Then the *covariant derivative* $\nabla\alpha$ is the element of $\Omega^1(M; P \times_\rho V)$ corresponding to

$$P_\omega^* d(\alpha) \in \Omega^1_{\rho,hor}(P; V)$$

.

Proposition 2.13. ∇ is a covariant derivative in the usual sense.

Lemma 2.14. If $\alpha \in \Omega^1_\rho(P; V)$ then

$$P_\omega^* d(\alpha) = d\alpha + \rho_*(\omega) \wedge \alpha.$$

Definition 2.15. The *curvature form* of ω is

$$\Omega = P_\omega^* d(\omega).$$

using the previous lemma we derive Cartan's second structure equation:

$$\Omega = d\omega + \frac{1}{2}[\omega \wedge \omega].$$

Proposition 2.16. For $\Omega^k_{\rho,hor}(P; V)$,

$$(P_\omega^* d)^2(\alpha) = \rho_* \Omega \wedge \alpha.$$

Since $\Omega \in \Omega^2_{Ad,hor}(P; \mathfrak{g})$ we get a corresponding element $K \in \Omega^2(M; End(P \times_\rho V))$.

Proposition 2.17. $K(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}$.

3 Clifford Bundles and Dirac Operators - Patrick Orson

3.1 Motivation

Historically the motivation for this subject comes from mathematical physics. P.A.M. Dirac was interested in finding a Lorentz-invariant wave equation $D\psi = \lambda\psi$ compatible with the Klein-Gordon equation

$$\sum_{i=0}^{3} \frac{\partial \psi}{\partial x_i} = \lambda\psi.$$

Causality required that D be first order in the 'time' variable. In essence, Dirac sought a first-order operator whose square was the Laplacian. So in n dimensions the ansatz becomes:

$$P = \sum \gamma_i \frac{\partial}{\partial x_i}$$

such that $P^2 = \Delta$, this is satisfied if and only if

$$\gamma_i^2 = -1 \text{ and } \gamma_i\gamma_j + \gamma_j\gamma_i = 0 \ \forall i \neq j.$$

This problem cannot be solved for $\gamma_i \in \mathbb{R}, \mathbb{C}$ except in the case $n = 1$, but can be solved by taking γ_i in a matrix algebra.

Example 3.1. Solutions for low dimensions

- $n = 1$: Set $\gamma_1 = i$ then $P = i\frac{\partial}{\partial x}$.
- $n = 2$: Take γ_i in the matrix algebra $M_2(\mathbb{C})$ and

$$P = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \frac{\partial}{\partial x} + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \frac{\partial}{\partial y}.$$

  Note that this makes the γ_i a representation of $\mathbb{C}$.

- $n = 3$: Now set the γ_i to be a representation of the quaternions in $M_2(\mathbb{C})$

$$\gamma_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \gamma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \gamma_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

In this approach we have taken elements of matrix algebras instead of simply real or complex coefficients and hence changed the space on which we were acting. If before we were interested in finding a square root of the Laplacian acting on functions $f : \mathbb{R}^n \to \mathbb{R}$ or $\mathbb{C}$, and this must be adjusted. Generally, the unital algebra in which $\gamma_i^2 = -1$ and $\gamma_i\gamma_j + \gamma_j\gamma_i = 0 \ \forall i \neq j$ is called the Clifford algebra Cl_n and hence we are looking for vector spaces V and representation $\kappa : Cl_n \to \mathrm{End}(V)$.

3.2 Clifford Modules

Definition 3.2. Let (V, q) be a vector space over $F = \mathbb{R}$ or $\mathbb{C}$ with a quadratic form. Then a Clifford algebra $Cl(V, q)$ is a unital algebra with a map $\phi : V \to Cl(V, q)$, such that $\phi(v)^2 = -q(v)1$ for each $v \in V$, that is universal among such algebras. That is, if $\psi : V \to (A)$ is a function with $\psi(v)^2 = -q(v)1$ for each $v \in V$, then there is a lift to a unique algebra homomorphism $\tilde{\psi}$

$$\begin{tikzcd} Cl(V,q) \arrow[r, dotted, "\tilde{\psi}"] & (A) \\ V \arrow[u, "\phi"] \arrow[ur, "\psi"'] & \end{tikzcd}$$

Proposition 3.3. $Cl(V, q)$ exists and is unique.

Proof. For existence, we may construct $Cl(V,q)$ it as follows. Take the tensor algebra $T(V)$ and quotient by the two-sided ideal $I(q)$ generated by elements of the form $v \otimes v + q(v) | v \in V$. One can check [**?**] that this algebra has the desired properties.

Uniqueness follows easily from the universal property. ∎

Easy facts to check:

- The map ϕ is injective. Hence we will usually omit the notation and consider elements of V to be just elements inside the Clifford algebra.
- If $e_1, ..., e_n$ is a basis of V then $\{e_1^{k_1} ... e_n^{k_n} | k_i = 0 \text{ or } 1\}$ is a basis of $Cl(V,q)$. Hence $\dim Cl(V,q) = 2^n$.
- As the characteristics of $\mathbb{R}$ and $\mathbb{C}$ are not 2, we have a polarisation identity telling us how to commute elements of the Clifford algebra.

$$\begin{aligned} vw + wv &= -2q(v,w) \\ &= -(q(v+w) - q(v) - q(w)) \end{aligned}$$

  In particular it will often be convenient to pick a pseudo-orthonormal basis of V so that basis elements of V anti-commute in $Cl(V,q)$.

What do the low-dimensional (real) Clifford algebras actually *look like*? It turns out they are very familiar objects.

Example 3.4. Let q be a non-degenerate form over $\mathbb{R}^n$ with signature (r,s) and let $e_1, ..., e_n$ be a pseudo-orthonormal basis. Denote the corresponding Clifford Algebra $Cl(V,q) = Cl_{r,s}$. In the special case $s = 0$, denote the algebra Cl_n.

- $n = 1$: Cl_1 is a unital algebra with basis $1, e$ such that $e^2 = -1$. This must be $\mathbb{C}$. The opposite form gives us $Cl_{0,1}$ and it is easily checked that this is $\mathbb{R} \oplus \mathbb{R}$.
- $n = 2$: Cl_2 has basis $1, e_1, e_2, e_1e_2$. If we form a map to the quaternions $V \to \mathbb{H}$, with $e_1 \mapsto i$ and $e_2 \mapsto j$ then the associated algebra homomorphism $Cl_2 \to \mathbb{H}$ is in fact an isomorphism (*check*).
- $n = 3$: $Cl_3 \cong \mathbb{H} \oplus \mathbb{H}$...

Proposition 3.5. There is a complete classification of the real Clifford algebras and it is (in some sense) modulo 8 in each argument of the signature. All Clifford algebras are isomorphic to matrix rings over $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, or sums thereof. See [**?**], Table II.

3.3 Dirac Operators

We will now restrict to the case where q comes from an inner product $(\ ,\ )$, hence the signature will be $(n,0)$ and we will drop reference to the dimension where this causes no confusion. Moreover it will be convenient to consider complexified Clifford algebras $Cl(V) \otimes_{\mathbb{R}} \mathbb{C}$.

Definition 3.6. A *Clifford module* S for a real inner product space V to be a vector space which is a left module over $Cl(V) \otimes \mathbb{C}$. In other words, we have a *Clifford multiplication* on elements of S by elements in the Clifford algebra, that returns another element of S. I will denote the Clifford multiplication by '$\cdot$'. (One could also think of this as a representation $c : \mathrm{Cl}(V) \otimes \mathbb{C} \to \mathrm{End}(S)$.)

The Clifford multiplication can also be performed on elements $s \in C^\infty(V,S)$, the S-valued functions on V.

Definition 3.7. The *Dirac operator* of the Clifford module is:

$$Ds = \sum_i e_i \cdot (\partial_i s).$$

This has the property we have been trying to develop:

$$D^2 s = \sum_{i,j} e_j \cdot \partial_j (e_i \cdot \partial_i s) = \sum_{i,j} e_j \cdot (e_i \cdot (\partial_j \partial_i s)) = -\sum_i \partial_i^2 s.$$

We now wish to transfer this construction to vector bundles on a Riemannian manifold (M, g). Each tangent space $T_m M$ has an inner product and so is a natural choice of V for the construction above.

Definition 3.8. Let S be a smooth vector bundle over M such that the fibres S_m are Clifford modules over $Cl(T_m M) \otimes \mathbb{C}$. S is a *Clifford bundle* if it is equipped with a Hermitian metric h and compatible connection ∇^S satisfying the following properties:

1. Skew adjointness: $h(v \cdot s_1, s_2) + h(s_1, v \cdot s_2) = 0$ for each $v \in TM$ and $s_1, s_2 \in S$.
2. Compatibility with Levi-Cevita:

$$\nabla_X^S (Y \cdot s) = (\nabla_X Y) \cdot s + Y \cdot \nabla_X^S s$$

for any vector fields X, Y and $s \in S$.

Remark 3.9. The condition (1) will ensure that the Dirac operator is self adjoint. The condition (2) is a natural thing to ask for in the sense that if we were to construct both TM and S as associated bundles of same principal bundle P, the connections are both push-forwards of the same connection on P.

Definition 3.10. The *Dirac operator* D of a Clifford bundle is the first order differential operator on $\Omega^0(S)$ defined by the composition

$$\Omega^0(S) \xrightarrow{\nabla^S} \Omega^1(S) \xrightarrow{metric} \Gamma(TM \otimes S) \xrightarrow{Clifford} \Omega^0(S)$$

.

So if we fix a local orthonormal frame $e_1, ..., e_n$ for TM, we get

$$s \mapsto \sum e^i \otimes \nabla_i^S s \mapsto \sum e_i \otimes \nabla_i^S s \mapsto \sum e_i \cdot \nabla_i^S s$$

i.e. locally

$$Ds = \sum e_i \cdot \nabla_i^S s.$$

Remark 3.11. For brevity, and where it causes no confusion, I will usually drop reference to which metric and connection I am using.

Proposition 3.12. (Weitzenbock Formula)

$$D^2 s = \nabla^* \nabla s + K s,$$

where ∇^* is the formal adjoint of ∇, and K is the 'Clifford contraction' of the curvature 2-form on S given by $K = \sum_{i<j} c(e_i) c(e_j) K(e_i, e_j)$.

Definition 3.13. A frame over a vector bundle $V \to M$ near a point $m \in M$ is called *synchronous* at m with respect to local co-ordinates near m if all the connection coefficients vanish at m. Such a local frame can always be chosen by taking a frame above m with this property and parallel transporting it along radial lines.

Pick a local frame $e_1, ..., e_n$ of TM that is synchronous at $m \in M$ (with respect to local co-ordinates) and let $s \in C^\infty(S)$. Then, as the connection coefficients vanish, $\nabla_i e_j = 0$ and $[e_i, e_j] = 0$ for all i, j. Then at m:

$$\begin{aligned} D^2 s &= \sum_{i,j} e_i \cdot \nabla_i (e_j \cdot \nabla_j s) \\ &= \sum_{i,j} e_i \cdot e_j \cdot \nabla_i \nabla_j s \\ &= -\sum_i \nabla_i^2 s + \sum_{i<j} e_i \cdot e_j \cdot (\nabla_i \nabla_j - \nabla_j \nabla_i) s \end{aligned}$$

The first term is Laplacian-like and by a calculation in local co-ordinates it can be verified that it is indeed $\nabla^*\nabla$ (see [**?**]). The curvature 2-form is given in local co-ordinates by $K(e_i, e_j) = \nabla_i\nabla_j - \nabla_j\nabla_i - \nabla_{[\partial_i,\partial_j]}$. So at m the second term in the equation above is the curvature.

Proposition 3.14. D is self adjoint.

3.4 Examples of Clifford Bundles

3.4.1 Regular Representation

Fix a vector space V with an inner product. $Cl(V) \cong \bigwedge^* V$ as vector spaces (but not as algebras). Further, $\bigwedge V \cong \bigwedge^* V$ so there is an isomorphism $\phi : Cl(V) \to \bigwedge^* V$. We can exploit this fact to build a Clifford module. $Cl(TM) \otimes \mathbb{C}$ acts on itself on the left by multiplication. Hence $S = \bigwedge^* V \otimes \mathbb{C}$ is a Clifford module with representation:

$$c : \bigwedge^* TM \to \mathrm{End}_{\mathbb{C}}(\bigwedge^* TM)$$

given by $c(v)w = \phi(v \cdot w)$. We can express this more concretely in terms of the interior and exterior products in $\bigwedge^* TM$ (see [**?**] for details of the interior product ι).

Claim 3.15. Let e, $\omega \in \Omega^k(M)$, then $c(e)w = e \wedge \omega - \iota(e)\omega$.

Proof. Calculation. ∎

To check that this forms a Clifford bundle we still need the compatibility conditions for the metric and connection. These are shown in [**?**]. What is the associated Dirac operator?

$$\begin{aligned} D\omega &= \sum_i c(e_i)\nabla_i\omega \\ &= \sum_i e_i \wedge \nabla_i\omega + \iota(e_i)\nabla_i\omega \\ &= d\omega + d^*\omega \end{aligned}$$

(the de-Rham operator) and

$$D^2 = dd^* + d^*d$$

.

3.4.2 Spin Representation

Suppose V is a vector space with $\dim_{\mathbb{R}} V = 2m$ and $J : V \to V$ is a complex structure. On top of this, complexify V. Then we may always decompose

$$V \otimes_{\mathbb{R}} \mathbb{C} = P \oplus Q$$

as the $\pm i$ eigenspaces of J. P and Q are the maximal *isotropic* subspaces (i.e. $(p_1, p_2) = (q_1, q_2) = 0$ for $p_i \in P$ and $q_i \in Q$).

Our intended Clifford module is $\bigwedge^* P$ and our Clifford algebra is $Cl(V) \otimes \mathbb{C}$. Let $p + q \in V \otimes \mathbb{C}$, then action can be defined as:

$$(p + q) \cdot x = \sqrt{2}(p \wedge x - \iota(q)x).$$

This extends to an action of $Cl(V) \otimes \mathbb{C}$.

When can we extend this process to bundles? We would require at least a $2m$-dimensional Riemannian manifold with an almost complex structure $J : TM \to TM$ and compatible metric. One can check that if the manifold M is in fact complex itself, then with a compatible Hermitian metric and appropriate connection, $\bigwedge^* T_{\mathbb{C}}M$ (the bundle form of P) is a Clifford bundle.

Theorem 3.16. If M is Kähler then $D = \sqrt{2}(\bar{\partial} + \bar{\partial}^*)$.

5 Spectral Properties of the Dirac Operator - Mark Powell

5.1 Outline

Let S be a Clifford bundle on M compact. Let D be such that $D^2 = \nabla^*\nabla + B$ where B is some first order operator. **We wish to show that e^{-tD^2} makes sense as an operator.**

D is an operator on $C^\infty(S)$ but we want to work with the properties of a Hilbert space.

IDEA: We pass back and forth between $C^\infty(S)$ and $L^2(S)$.

$$\begin{tikzcd} D : C^\infty(S) \arrow[r] \arrow[d] & C^\infty(S) \arrow[d] \\ \bar{D} : L^2(S) \arrow[r] & L^2(S) \end{tikzcd}$$

We will use nice properties of compact operators on Hilbert spaces to get a spectral decomposition and show that what we got was smooth after all.

Theorem 5.1. (Main Theorem of talk) There is an orthogonal decomposition

$$L^2(S) = \bigoplus_{\lambda \in \Lambda} L^2(S)_\lambda$$

into countably many finite-dimensional eigenspaces of smooth sections of D. Moreover $\Lambda \subseteq \mathbb{R}$ is discrete.

Consequently (using functional calculus), for $s \in L^2(S)$

$$s = \sum_{\lambda \in \Lambda} s_\lambda,$$

then we may *define*

$$e^{-tD^2}(s) = \sum_{\lambda \in \Lambda} e^{-t\lambda^2} s_\lambda.$$

5.2 Tools for the Proof

Definition 5.2. (Rough) The *Sobolev space* W^k is the space of sections of S whose first k derivatives are in $L^2(S)$.

$$C^\infty(S) \subseteq ... \subseteq W^k \subseteq ... \subseteq W^1 \subseteq W^0 = L^2(S)$$

More precisely W^k is the completion of $C^\infty(S)$ is $||\cdot||_k$ (the Sobolev k-norm).

$$||f||_k = \sum_{|\alpha| \leq k} ||\frac{\partial f}{\partial x^\alpha}||_{L^2}$$

Moreover, $\bigcap_k W^k = C^\infty(S)$ by the *Sobolev Embedding Theorem.*

Lemma 5.3. (Garding's Inequality) **5.14** [**?**]

$$||s||_1 \leq C(||s||_0 + ||Ds||_0)$$

for some $C > 0$. This is to say

$$\int (|s|^2 + |\nabla s|^2) \leq C(\int |s|^2 + \int |Ds|^2).$$

A generalisation of this is **5.16** [**?**]

$$||s||_{k+1} \leq C_k(||s||_k + ||Ds||_k).$$

So in particular, if $s \in W^k$ and $Ds = 0$ then $s \in W^{k+1}$. By induction this will allow us to pass to $s \in C^\infty(S)$.

Proposition 5.4. (**5.24** [**?**]) $\ker(\bar{D})$ comprises smooth sections.

Theorem 5.5. (Rellich) The inclusion $W^{k+1} \to W^k$ is a compact operator.

5.3 Proof of Main Theorem

Definition 5.6. Let G be the graph of D

$$G = \{(x, Dx) \in L^2(S) \oplus L^2(S) | x \in C^\infty(S)\}$$

and let $\bar{G}$ be the closure of G. $\bar{G}$ is also a graph (by Closed Graph Theorem). Define the operator $\bar{D}$ by this - it has domain W^1 due to Garding's Inequality.

$$\bar{D} : W^1 \to L^2(S)$$

Definition 5.7. Define $Q : L^2(S) \to W^1$ in the following way. Let $x \in L^2(S)$, then let Qx be such that $(Qx - x, \bar{D}(Qx)) \perp \bar{G}$, the perpendicular projection of $(x, 0)$ to $\bar{G}$.

- As Q is an orthogonal projection, it is self-adjoint.
- $||x||^2 = ||Qx||^2 + ||\bar{D}Qx||^2$, so Garding's Inequality implies that Q is bounded.
- Recalling Rellich's Theorem, we see that

  $$G : L^2(S) \to W^1 \to L^2(S)$$

  is compact and self adjoint.
- Spectral Theorem gives us

  $$L^2(S) = \bigoplus_\rho L^2(S)_\rho,$$

  a decomposition by eigenspaces of Q with discrete eigenvalues ρ, where $\rho \to 0$.

Claim 5.8. $G^\perp = JG$ for $J : (x, y) \mapsto (y, -x)$.

Proof. Let $(x, y) \in G^\perp$, then

$$\begin{array}{rrcll} & \langle (x,y), (s, Ds) \rangle & = & 0 & \forall s \in C^\infty(S) \\ \implies & \langle x, s \rangle + \langle y, Ds \rangle & = & 0 & \\ \implies & \langle x + Dy, s \rangle & = & 0 & \text{as } D = D^* \\ \\ \implies & \bar{D}y = -x & & & \end{array}$$

■

We may now proceed to prove the Main Theorem.

Proof.

$$\begin{aligned} L^2(S) \oplus L^2(S) &= \bar{G} \oplus \bar{G}^\perp \\ &= \bar{G} \oplus G^\perp \\ &= \bar{G} \oplus JG \\ &= \bar{G} \oplus J\bar{G} \end{aligned}$$

Let $x \in L^2(S)$ be an eigenvector of Q, then there exists $y \in L^2(S)$ such that

$$\begin{aligned} (x, 0) &= (Qx, \bar{D}Qx) + (-\bar{D}y, y) \\ &= \rho^2(x, \bar{D}x) + (\bar{D}y, y) \end{aligned}$$

So $(\rho^2 - 1)x = \bar{D}y$ and $y = -\rho^2\bar{D}x$. Let $\lambda^2 = (1 - \rho^2)/\rho^2$ and $z = -(1/\rho^2\lambda)y$. We may calculate that

$$\begin{aligned} \bar{D}x &= \lambda z \\ \bar{D}z &= \lambda x \end{aligned}$$

So $x \pm z$ are eigenvectors of $\bar{D}$ with eigenvalues $\pm\lambda$.

Hence $s_\lambda \in \ker(D - \lambda)$ and therefore $s_\lambda \in C^\infty(S)$, ■

8 The Lefschetz fixed point theorem - Spiros Adams-Florou

Though not absolutely essential for our studies of the Index Theorem, the Lefschetz formula is interesting enough in its own righ to be included because

- It is our first example of a topological invariant defined by elliptic operators
- The Lefschetz number of a smooth map may be expressed using a heat kernel in an analogous fashion to the index.
- It is a nice warm up for the index formula as the small-t computations are easier than those for the index theorem - we do not need a Getzler rescaling.
- We can actually exhibit the index as a Lefschetz number, but it is not so easy to compute via the formulae presented in this talk, as the fixed points are not simple.

Aim of the Talk:

- We define the Lefschetz number analytically in the context of Dirac complexes.
- We show that we can express it in terms of a heat kernel that seemingly depends on t but actually doesn't.
- We show that away from the fixed points the kernel decreases so rapidly that we don't get any contributions to the Lefschetz number.
- We calculate the contributions at the (simple) fixed points by looking at asymptotics for small t.
- We give a few examples.

Before we start we will need a few definitions:

Let H, H' be (seperable, infinite dimensional) Hilbert spaces with orthonormal bases (e_i), (e'_j). A bounded linear operator $A : H \to H'$ can be represented by an infinite matrix with coefficients

$$C_{ij}(A) = \langle Ae_i, e'_j \rangle.$$

Definition 8.1. An operator A is called a *Hilbert-Schmidt* operator if

$$\| A \|_{HS}^2 := \sum_{i,j} |c_{ij}(A)|^2 < \infty,$$

where $\| - \|_{HS}$ is called the Hilbert-Schmidt norm.

Remark 8.2.

1. Any Hilbert-Schmidt operator is necessarily compact.
2. The sum of two HS operators is again HS, and a HS operator composed (in either order) with a bounded operator is HS.

Definition 8.3. A bounded operator $T : H \to H$ is called *trace-class* if there exist HS operators A, B such that $T = AB$. The trace of such an operator is well defined by

$$Tr(T) := \langle A^*, B \rangle_{HS} = \sum_{i,j} \overline{C_{ij}}(A^*) c_{ij}(B)$$

and is independent of the choice of A and B.

Remark 8.4.

$$(\text{trace} - \text{class}) \subset (\text{Hilbert} - \text{Schmidt}) \subset (\text{compact}) \subset (\text{bounded}).$$

Proposition 8.5. Let T, B be bounded on H, then if either T is trace-class, or both T and B are HS, then TB and BT are trace-class and

$$\mathrm{Tr}(TB) = \mathrm{Tr}(BT).$$

Theorem 8.6. Let A be smoothing on $L^2(S)$, with kernel k. Then A is of trace-class, and

$$\mathrm{Tr}(A) = \int \mathrm{tr}k(m,m)\mathrm{vol}(m)$$

where $\mathrm{tr} : S_m \otimes S_m^* \to \mathbb{C}$ is the canonical trace on endomorphisms of the finite dimensional vector space S_m.

Let M be a manifold, $\phi : M \to M$ a smooth[1] map. We define the classical Lefschetz number by

$$L(\phi) := \sum_q (-1)^q \mathrm{tr}(\phi^* : H^q(M) \to H^q(M)).$$

This can be expressed as a sum over fixed points, so if $L(\phi) \neq 0$ we must have a fixed point. We will recover this formula later!

Now suppose we have a dirac complex (S, d) over M:

$$C^\infty(S_0) \xrightarrow{d} \dots \xrightarrow{d} C^\infty(S_k) \qquad\qquad S = \bigoplus_{i=0}^k S_i$$

then $\phi : M \to M$ induces $\phi^* : C^\infty(S) \to C^\infty(\phi^* S)$.

If (S, d) is the de Rham complex then there exists a natural bundle map $\zeta = \bigwedge^* T^*\phi : \phi^*(\bigwedge *T * M) \to \bigwedge *T * M$, but for a general dirac complex there may be no such map, so we must assume existence of ζ as part of our data.

Set $F = \zeta\phi^* : C^\infty(S) \to C^\infty(S)$.

Definition 8.7. If F is a chain map of complexes ($dF = Fd$), then we call (ζ, ϕ) a *geometric-endomorphism* of (S, d). In this case we define the Lefschetz number as

$$L(\zeta, \phi) := \sum_q (-1)^q \mathrm{tr}(F^* : H^q(S) \to H^q(S)).$$

In the case where our complex is the de Rham complex this definition gives back the classical definition.

By the Hodge theory of the last lecture $H^q(S)$ is represented by $\mathcal{H}^q$, the space of harmonic sections of S_q. Let $P_q : L^2(S_q) \to \mathcal{H}^q$ be orthogonal projection, then

$$\mathrm{tr}(F^* : H^q(S) \to H^q(S)) = \mathrm{Tr}(FP_q).$$

Lemma 8.8. If Δ_q denotes D^2 restricted to $C^\infty(S_q)$, then P_q is smoothing and as $t \to \infty$ the smoothing kernel of $e^{-t\Delta_q}$ tends to the smoothing kernel of P_q in the C^∞ topology.

Proof. See Roe 10.5. ∎

So we can write

$$L(\zeta, \phi) = \lim_{t\to\infty} \sum_q (-1)^q \mathrm{Tr}(Fe^{-t\Delta_q}).$$

Next we show that this does not actually depend on $t > 0$.

Proposition 8.9. For all $t > 0$,

$$\sum_q (-1)^q \mathrm{Tr}(Fe^{-t\Delta_q}) = L(\zeta, \phi).$$

Proof. We show that the LHS is constant in t. Differentiating the LHS we get

$$\sum (-1)^{q+1} \mathrm{Tr}(F(dd^* + d^*d)e^{-t\Delta_q}).$$

[1] We do not require smoothness to define the classical Lefschetz number, but for the rest of the talk our map ϕ will be smooth.

We will see that the terms from dd^* cancel in pairs with those from d^*d.

$$\begin{aligned} dF = Fd \Rightarrow \mathrm{Tr}(Fdd^*e^{-t\Delta_q}) &= \mathrm{Tr}(dFd^*e^{-t\Delta_q}) \\ &= \mathrm{Tr}(\underbrace{dFd^*e^{-t\Delta_q/2}}_{\text{Bounded}}\ \underbrace{e^{-t\Delta_q/2}}_{\text{trace-class}}) \\ &= \mathrm{Tr}(\underbrace{e^{-t\Delta_q/2}d}_{\text{trace-class}}\underbrace{Fd^*e^{-t\Delta_q/2}}_{\text{Bounded}}) \\ &= \mathrm{Tr}(Fd^*e^{-t\Delta_q}d) \\ &= \mathrm{Tr}(Fd^*de^{-t\Delta_{q-1}}) \end{aligned}$$

as $\Delta_q d = d\Delta_{q-1}$. In the above we have used Proposition 8.5 twice. For an explanation as to why the operators above are bounded please refer to remark 10.4 in [**?**]. Thus the terms cancel in pairs. ■

Next we see that away from the fixed points of ϕ we get no contribution to $L(\zeta, \phi)$:

Proposition 8.10. If ϕ has no fixed points, then $L(\zeta, \phi) = 0$.

Proof. We examine the asymptotic expansion for

$$L(\zeta, \phi) = \sum_q (-1)^q \mathrm{Tr}(Fe^{-t\Delta_q})$$

for small t. We know from Michael's lecture that $e^{-t\Delta_q}$ has a Schwartz kernel, i.e.

$$(e^{-t\Delta_q}s)(m_1) = \int_M k_t^q(m_1, m_2)s(m_2)dvol(m_2).$$

$Fe^{-t\Delta_q}$ is smoothing, so by Theorem 8.12 of [**?**] it is trace-class, and

$$\mathrm{Tr}(Fe^{-t\Delta_q}) = \int_M \mathrm{tr}({}_1\zeta k_t^q(\phi(m), m))dvol(m),$$

where ${}_1\zeta$ denotes ζ acting on the first variable in $S \boxtimes S^*$.

If ϕ has no fixed points then the graph of ϕ does not intersect the diagonal in $M \times M$. By Michael's lecture, the kernel $k_t^q(m_1, m_2) \to \delta_{m_2}$ as $t \to 0$. There is a non-zero distance between $\phi(m)$ and m so we can make $k_t^q(\phi(m), m)$ as small as we like by taking t sufficiently small. We see that $k_t^q(\phi(m), m) \to 0$ uniformly in m as $t \to 0$. So the result follows. ■

Example 8.11. Let ϕ be any holomorphic automorphism of $\mathbb{P}^n$, then ϕ has a fixed point:

M is Kähler, so its Dolbeault complex is Dirac.

$$H^q_{\bar\partial}(M) = \begin{cases} \mathbb{C}, & q = 0 \\ 0, & q > 0 \end{cases}$$

and $\phi^* = \mathrm{id} : H^0_{\bar\partial} \to H^0_{\bar\partial}$. Thus $L(\phi) = 1$, so ϕ has a fixed point.

So we have shown that away from the fixed points we do not get a contribution to the Lefschetz number, next we show that at simple fixed points we do get a contribution and we compute a formula for it.

Let $T_m\phi$ denote the endomorphism $T_mM \to T_{\phi(m)}M = T_mM$ induced by ϕ.

Definition 8.12. The fixed point m is *simple* if $\det(1 - T_m\phi) \neq 0$, or equivalently if the graph of ϕ cuts the diagonal transversally at (m, m).

If all the fixed points are simple then they must be isolates, and since M is compact we can only have finitely many fixed points.

Lemma 8.13. Let T be an $n \times n$ matrix with simple fixed points, then $\forall t > 0$,

$$\frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{\frac{-|x-Tx|^2}{4t}} d^n x = \frac{1}{|\det(1-T)|}.$$

Proof. Let $A = (1-T)(1-T^*)$, so $|x - Tx|^2 = (Ax, x)$. Then if $\lambda_1, \ldots, \lambda_n$ are eigenvalues of A, making an orthogonal change of coordinates on $\mathbb{R}^n$ to the eigenbasis the result follows by straight-forward integration. ∎

Remark 8.14. The integral we have just computed is of the leading term in the asymptotic expansion for the heat kernel as given in Michael's lecture. Hence the RHS will appear in the formula for the contributions at the fixed points.

Theorem 8.15 (ATIYAH-BOTT)**.** Let (ζ, ϕ) be a geometric endomorphism of a Dirac complex (S, d) having only simple fixed points, then

$$L(\zeta, \phi) = \sum_{\phi(m)=m} \sum_{q=0}^{n} \left(\frac{(-1)^q \mathrm{tr}(\zeta_q(m))}{|\det(1 - T_m\phi)|} \right).$$

Very sketchy proof. Since the kernel rapidly decreases away from fixed points we need only integrate over arbitrarily small neighbourhoods of the fixed points.

In geodesic co-ordinates with the origin at a fixed point we have

$$\begin{aligned} \zeta_q(x) &= \zeta_q(0) + O(|x|) \\ \phi(x) &= T_0\phi x + O(|x|^2) \\ g(x) &= 1 + O(|x|) \end{aligned}$$

where $g = \det(g_{ij})$.

We now use the asymptotic expansion

$$k_t^q(m', m) = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{\frac{-d(m^Prime, m)^2}{4t}} (\Theta_0(m', m) + O(t)) + O(t)$$

where $\Theta_0(m, m) = 1$.

The fact that the fixed points are simple gives us that

$$\det(1 - T_0\phi) \neq 0 \Rightarrow \exists \delta > 0 \; s.t. \; |x - T_0\phi x|^2 \geq \delta |x|^2.$$

Also,

$$\begin{aligned} d(\phi(x), x)^2 &= |x + T_0\phi x|^2 + O(|x|^2) \\ \Theta_0(\phi(x), x) &= 1 + O(|x|) \end{aligned}$$

We put all this together to show that

$$|_1\zeta_q(x) k_t^q(\phi(x), x)\sqrt{g}(x) - \frac{\zeta_q(0)}{(4\pi t)^{\frac{n}{2}}} e^{\frac{-|x - T_0\phi x|^2}{4t}}| \leq \text{something of order } t^{\frac{1}{2}} \text{ in } L^1.$$

This in turn shows that

$$\mathrm{Tr}(Fe^{-t\Delta_q}) \to \sum_{\phi(m)=m} \frac{\mathrm{tr}(\zeta_q(m))}{|\det(1 - T_m\phi)|}$$

and the result follows. ∎

Example 8.16. Now let's apply this to the de Rham complex. For the de Rham complex

$$\zeta_q = \bigwedge^q T^*\phi : \phi^* S_q \to S_q.$$

For any linear T,

$$\sum_q (-1)^q \mathrm{tr}(\bigwedge^q T) = \sum_q (-1)^q \sum_{i_1 < \ldots < i_q} \lambda_{i_1} \ldots . \lambda_{i_q} = \prod (1 - \lambda_i) = \det(1 - T).$$

Thus,

$$\begin{aligned} L(\zeta,\phi) &= \sum_{\phi(m)=m} \frac{\sum(-1)^q \mathrm{tr}(\bigwedge^q T_m^*\phi)}{|\det(1-T_m\phi)|} \\ &= \sum_{\phi(m)=m} \frac{\det(1-T_m^*\phi)}{|\det(1-T_m\phi)|} \\ &= \sum_{\phi(m)=m} \operatorname{sgn}\det(1-T_m\phi) \end{aligned}$$

thus recovering the original Lefschetz theorem.

9 The Index Problem - José Figueroa-O'Farrill

Definition 9.1. A $Cl(V)$-module W is *graded* if $W = W^+ \oplus W^-$ and for all $v \in V$

$$v : W^{\pm} \to W^{\mp}.$$

Now for (M, g), a Clifford bundle $S \to M$ is *graded* if

- $\forall p \in M$, S_p is a graded $Cl(T_pM)$-module.
- $S = S^+ \oplus S^-$ such that the direct sum is perpendicular with respect to the metric and compatible with the connection.

Equivalently there is a *grading operator/element*

$$\epsilon : S \to S$$

such that ϵ is self adjoint, $\epsilon^2 = Id.$ and $\epsilon|_{S_\pm} = \pm Id.|_{S_\pm}$, $\nabla\epsilon = 0$ and $\forall X \in \mathfrak{X}(M)$

$$c(X)\epsilon = -\epsilon c(X).$$

Remark 9.2. $C^\infty(S) = C^\infty(S^+) \oplus C^\infty(S^-)$ and $L^2(S) = L^2(S^+) \oplus L^2(S^-)$. This makes the space of bounded operators $\mathcal{B}(L^2(S))$ into a *superalgebra*.

Definition 9.3. A *superalgebra* is a graded algebra $A = A^0 \oplus A^1$ where $A_iA_j = A_{i+j \pmod 2}$

Suppose there is an element ϕ acting with respect to the grading:

$$\phi = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Now assume ϕ is *trace class*, then the supertrace is

$$\operatorname{Str}\phi = \operatorname{tr} a - \operatorname{tr} d \qquad (= \operatorname{tr}(\epsilon\phi)).$$

Proposition 9.4. If ψ is bounded then the *super commutator* vanishes:

$$\operatorname{Str}[\phi, \psi]_S = 0.$$

Note that if ϕ and ψ have definite parities then $[\phi, \psi]_S = \phi\psi - (-1)^{|\phi||\psi|}\phi\psi$.

Proof. Check cases. ∎

Suppose A is a smoothing operator on $L^2(S)$ with smoothing kernel

$$k(x, y) \in C^\infty(S \boxtimes S^*).$$

Definition 9.5.

$$\operatorname{Str} A = \int_M \operatorname{Str} k(x, x)\, \operatorname{dvol}(x)$$

where $k(x, x) \in S_x \otimes S_x^*$.

Now let (M^{2m}, g) be oriented, ω a volume form and $\theta^1, \theta^2, ..., \theta^{2m}$ be a local orthonormal coframe such that $g = \sum_1^{2m} (\theta^i)^2$ and $\omega = \theta^1 \wedge ... \wedge \theta^{2m}$. Then

$$c(\omega) : C^\infty(S) \to C^\infty(S)$$

and a calculation shows that $c(\omega)^2 = (-1)^m$.

Definition 9.6. The *canonical grading element* is $\epsilon_0 = i^m\omega$. (Check that this is a grading element.)

FACT: If Δ is the spin representation then

$$\begin{aligned} S_x &\cong \Delta \oplus ... \oplus \Delta \qquad \text{(as a } Cl(T_xM)\text{-module)} \\ &\cong \Delta \otimes V \end{aligned}$$

where V is the auxilliary vector space $\mathrm{Hom}_{Cl}(\Delta, S_x)$. Hence

$$\begin{aligned} \mathrm{End}(S_x) &\cong \mathrm{End}(\Delta) \otimes \mathrm{End}(V) \\ &\cong Cl(T_xM) \otimes \mathrm{End}_{Cl}(S_x) \end{aligned}$$

Now let $a \in \mathrm{End}(S)$ and write $a = c \otimes F$. The point is that $S^{\pm} = \Delta^{\pm} \otimes V$ so that $\epsilon = \epsilon_0 \otimes Id_V$. Hence $\mathrm{Str}\, a = \mathrm{Str}_{Cl}(c) \cdot \mathrm{tr}^{S/\Delta} F$. So we want to compute:

$$\mathrm{Str}_{Cl}(c) = \mathrm{tr}_{Cl}(i^m \omega c).$$

FACT: $\mathrm{tr}_{Cl}\, c = 0$ unless $c \propto Id$.

So, recalling the vector spcae isomorphism $Cl(V) \cong \bigwedge^* V$,

$$c = c_0 Id + \sum_i c_i e_e + \sum_{i<j} c_{ij} e_{ij} + ... + c_\omega \omega$$

where e are the basis elements. Hence $\mathrm{tr}_{Cl}\, c = c_0 \,\mathrm{tr}\, Id = c_0 2^m$ and

$$\mathrm{tr}_{Cl}(\omega c) = c_\omega (-1)^m 2^m.$$

9.1 Dirac Operator

Recall

$$D = \sum_{i=1}^{2m} c(e_i) \nabla_{e_i}$$

and $D\epsilon_0 = -\epsilon D$ so that

$$D = \begin{pmatrix} 0 & D_- \\ D_+ & 0 \end{pmatrix}$$

where $D_\pm = D|_{S^\pm} : C^\infty(S^\pm) \to C^\infty(S^\mp)$ and as D is self adjoint, $D_- = D_+^*$. Note as well that

$$D^2 = \begin{pmatrix} D_- D_+ & 0 \\ 0 & D_+ D_- \end{pmatrix}.$$

So we obtain a complex

$$C^\infty(S^+) \xrightarrow{D_+} C^\infty(S^-)$$

with Euler characteristic

$$\begin{aligned} \chi &= \dim\ker D_+ - \dim\mathrm{coker}\, D_+ \\ &= \dim\ker D_+ - \dim\ker D_- \\ &= \mathrm{Ind}\, D \end{aligned}$$

Note that we can consider the index of D as a supertrace by denoting by P the orthogonal projection $P : C^\infty(S) \to \ker D$. Then

$$\mathrm{Ind}\, D = \mathrm{Str}\, P.$$

9.2 Schwarz Function

Denote by $\mathcal{S}(\mathbb{R}^+)$ the space of Schwarz functions on $\mathbb{R}^+$. Let $f \in \mathcal{S}(\mathbb{R}^+)$ and $f(0) = 1$.

Proposition 9.7. $\operatorname{Ind} D = \operatorname{Str} f(D^2)$

Proof. From the discreteness of the spectrum of D, there is a minimal non-zero eigenvalue $\lambda_{\min}$ of D^2 and hence we may dilate F so that $f(\lambda) = 0$ for $\lambda < \lambda_{\min}$. Moreover, we may decompose any Schwarz function into the part with compact support $f(\lambda) = f_{\mathrm{cs}}(\lambda) + g(\lambda)$. So it is enough to show that if $g \in \mathcal{S}(\mathbb{R}^+)$ and $g(0) = 0$ then $\operatorname{Str} g(D^2) = 0$. We can write

$$\begin{aligned} g(\lambda) &= \lambda h(\lambda) \\ &= \lambda h_1(\lambda) h_2(\lambda) \end{aligned}$$

for some Schwarz functions h_1 and h_2. So

$$\begin{aligned} g(D^2) &= D^2 h_1(D^2) h_2(D^2) \\ &= \frac{1}{2}[D h_1(D^2), D h_2(D^2)]_S \end{aligned}$$

Hence $\operatorname{Str} d(D^2) = 0$.

■

Example 9.8. (The McKean Singer Formula)

$$\operatorname{Ind} D = \operatorname{Str}(e^{-tD^2})$$

which is independent of t as

$$\begin{aligned} \frac{d}{dt} \operatorname{Str}(e^{-tD^2}) &= -\operatorname{Str}(D^2 e^{-tD^2}) \\ &= -\operatorname{Str}(\frac{1}{2}[D e^{-tD^2/2}, D e^{-tD^2/2}]_S) \\ &= 0 \end{aligned}$$

So we can try an asymptotic expansion:

$$\operatorname{Str}(e^{-tD^2}) \sim \int_M \operatorname{Str} k_t(x,x) \operatorname{dvol}(x)$$

where

$$k_t(x,y) \sim \frac{1}{(4\pi t)^m} \exp\left(-\frac{d(x,y)^2}{4t}\right) \sum_{j \geq 0} t^j \Theta_j(x,y)$$

and $\Theta_j(x,y) \in S_x \otimes S_y^*$. Hence

$$\operatorname{Str}(e^{-tD^2}) \sim \frac{1}{(4\pi t)^{n/2}} \sum_j t^j \int_M \operatorname{Str} \Theta_j(x) \operatorname{dvol}(x).$$

So this must depend only on the term independent of t, i.e. $j = n/2$. So in fact

$$\operatorname{Ind} D = \frac{1}{(4\pi t)^{n/2}} \int_M \operatorname{Str}(\Theta_{n/2}) \operatorname{dvol}(x).$$

Remark 9.9. We already have nice relationships for coverings. If $\widetilde{M} \to M$ a k-fold covering then as this can be taken to be isometric and local we have

$$\int_{\widetilde{M}} \mathrm{Str}(\widetilde{\Theta_{n/2}})\widetilde{\mathrm{dvol}} = k \int_M \mathrm{Str}(\Theta_{n/2})\,\mathrm{dvol},$$

and so

$$\mathrm{Ind}\,\widetilde{D} = (\mathrm{Ind}\,D) \cdot k.$$

This shows that a local formula behaves nicely on the topological side.

The Index Problem

Show that $\mathrm{Ind}\,D$ is independent of homotopy, i.e. write it in terms of charcteristic classes of TM and S.

9.3 Filtered Algebras

Definition 9.10. $A = \bigoplus_{m \geq 0} A^m$ is a *graded algebra* if $A^m A^{m'} \subseteq A^{m+m'}$.

Example 9.11. $\bigotimes V$, $\bigwedge V$, $\mathrm{Sym}\,V$, $\mathbb{C}[t]$,... But not $Cl(V)$.

Definition 9.12. If A is such that $0 \subseteq ... \subseteq A_{m-1} \subseteq A_m \subseteq ... \subseteq A$ with $A_m A_{m'} \subseteq A_{m+m'}$, it is a *filtered algebra.*

Example 9.13.

- $Cl(V)$
- $\mathcal{D}(M) =$ differential operators on $C^\infty(M)$. $X \in \mathfrak{X}(M)$ and $f \in C^\infty(M)$ generate $\mathcal{D}(M)$ with $\deg f = 0$ and $\deg X = 1$.
- $\mathcal{D}_m =$ operators of order $\leq m$.

Definition 9.14. Suppose A is a filtered algebra and G is a graded algebra. A *symbol map* $\sigma : A \to G$ is a family of linear maps $\sigma_m : A_m \to G^m$ such that

1. $\sigma_m(a) = 0$ if $a \in A_{m-1}$.
2. If $a \in A_m$, $b \in A_{m'}$ then $\sigma_{m+m'}(ab) = \sigma_m(a)\sigma_{m'}(b)$.

Definition 9.15. Let A be a filtered algebra and define the *associated graded algebra* $G(A)$ by $G(A)_m = A_m/A_{m-1}$ and the *universal symbol map* σ by linear maps:

$$\begin{aligned} \sigma_m = \mathrm{gr}_m : A_m &\to A_m/A_{m-1} \\ a &\mapsto [a] \end{aligned}$$

Example 9.16.

$$\begin{aligned} Cl(V) &\xrightarrow{\mathrm{gr}} \bigwedge^* V \\ e_i e_j &\mapsto e_i \wedge e_j \\ \frac{1}{2}(e_i e_j - e_i e_j) + \frac{1}{2}(e_i e_j + e_i e_j) &\mapsto \frac{1}{2}(e_i e_j - e_i e_j) \end{aligned}$$

Example 9.17. For V a vector space define a graded algebra $C(V)$ to be the constant coefficient differential operators on $C^\infty(V)$.

Now consider $C(TM)$ with fibre $C(T_pM)$. $C^\infty(C(TM))$ is a graded algebra. Define $\sigma : \mathcal{D}(M) \to C^\infty(C(TM))$ by fixing $p \in M$ and local co-ordinates centred at p

$$T := \Sigma_{|\alpha|\leq m} c_\alpha(x) \frac{\partial^\alpha}{\partial x^\alpha}$$

$$\sigma_{m,p}(T) := \Sigma_{|\alpha|=m} c_\alpha(0) \frac{\partial^\alpha}{\partial x^\alpha}$$

This does not depend on the choice of co-ordinates (check this) and for each p this defines a symbol map (check this). So as p varies we get a symbol map

$$\sigma : \mathcal{D}(M) \to C^\infty(C(M)).$$

Example 9.18. Let $f \in C^\infty(M)$ and $X \in \mathfrak{X}(M)$, then define $\sigma_0(f) = f$ and $\sigma_1(X) = X$ i.e. multiplication by $f(p)$ and action of $\frac{\partial}{\partial X(p)}$.

Example 9.19. To filter $\mathrm{End}(S)$, split it as $\mathrm{End}(S) = Cl(TM) \otimes \mathrm{End}_{Cl}(S)$ where we use the Clifford filtration on $Cl(TM)$ and set $\mathrm{End}_{Cl}(S)$ to filter degree 0.

$$\sigma : \mathrm{End}(S) \to (\bigwedge TM) \otimes \mathrm{End}_{Cl}(S)$$

Definition 9.20. Define $\mathcal{D}(S)$ to be the differential operators on $C^\infty(S)$ generated by ∇_X for $X \in \mathfrak{X}(M)$, $c(X)$ and $F \in C^\infty(\mathrm{End}_{Cl}(S))$.

Getzler degree	Operator
1	∇_X
1	$c(X)$
0	F

We want to define a symbol

$$\sigma : \mathcal{D}(S) \to C^\infty(??? \otimes \bigwedge TM \otimes \mathrm{End}_{Cl}(S))$$

Definition 9.21. For a vector space V define $\mathcal{P}(V)$ to be the polynomial coefficient differential operators on $C^\infty(V)$. It is graded:

$$\deg(x^\alpha \frac{\partial^\beta}{\partial x^\beta}) = |\beta| - |\alpha|.$$

Let $\mathcal{R}_p \in \bigwedge^2 T_p^*M \otimes \mathrm{End}(T_pM)$ be the Riemannian tensor. By evaluating at $X_p \in T_pM$ and contracting using the metric with some $v \in T_pM$ we get

$$\begin{array}{rcl} T_pM & \to & \Lambda^2 T_p^*M (\cong \Lambda^2 T_pM) \\ v & \mapsto & (\mathcal{R}_p(\,\cdot\,,\,\cdot\,)X_p, v) \end{array}$$

So for $X \in \mathfrak{X}(M)$ we have

$$(\mathcal{R}X, \cdot\,) \in C^\infty(\mathcal{P}(TM) \otimes \Lambda^2 TM).$$

Relative to a local orthonormal frame :

$$\begin{aligned} (\mathcal{R}e_i, \cdot\,) &= \frac{1}{2}\Sigma(\mathcal{R}(e_k, e_l)e_i, e_j)x^j e_k \wedge e_l \\ &= \frac{1}{2}\Sigma(\mathcal{R}(e_i, e_j)e_k, e_l)x^j e_k \wedge e_l \end{aligned}$$

Definition 9.22. The Getzler symbol for the filtered algebra $\mathcal{D}(S)$ is

$$\sigma : \mathcal{D}(S) \to C^\infty(\mathcal{P}(TM) \otimes \Lambda TM \otimes \mathrm{End}_{Cl}(S))$$

defined on the generators by

$$\begin{aligned} \sigma_0(F) &= F, \\ \sigma_1(c(X)) &= e(X) \qquad \text{(exterior product)} \\ \sigma_1(\nabla_X) &= \partial_X + \frac{1}{4}(\mathcal{R}X, \cdot\,) \end{aligned}$$

Roe shows this is well defined.

Example 9.23. If $D = \Sigma_i c(e_i)\nabla_i$ then

$$\sigma_2(D) = \Sigma e(e_i)\frac{\partial}{\partial X_i} \sim \Sigma dx_i \wedge \partial_i = d_{TM}.$$

Using the Weitzenbock formula, $D^2 = \nabla^*\nabla + \frac{1}{4}\kappa + F^s$, where in local coordinates

$$F^s = \frac{1}{2}\Sigma F^s(e_i, e_j)e_i \wedge e_j \in \Lambda^2 T^*M \otimes \mathrm{End}_{Cl}(S)$$

$$\nabla^*(dx^i \otimes s_i) = -\Sigma g^{ij}(\nabla_i s_j - \Gamma^k_{ij} s_k)$$

and

$$\nabla^*\nabla = -\Sigma g^{ij}(\nabla_i\nabla_j - \Gamma^k_{ij}\nabla_k)$$

D^2 has Getzler degree 2 and

$$\begin{aligned}\sigma_2(D^2)_p &= -\Sigma_i\sigma_1(\nabla_i)^2 + \sigma_0(F^s)\\ &= -\Sigma_i(\frac{\partial}{\partial X_i} - \frac{1}{4}\Sigma R_{ij}X^j)^2 + F^s\end{aligned}$$

We want to define σ on smoothing operators as well, such as the one defined by $k_t(p,q)$. Let $s \in C^\infty(S \boxtimes S^*)$, fix $q \in M$ and define a map $p \mapsto s(p,q)$, choose geodesic co-ordinates x^i based at q. Take $s_\alpha \in C^\infty(S \otimes S^*_q)$ to be synchronous sections i.e. $\nabla_{\frac{\partial}{\partial r}} s_\alpha = 0$.

$$s_q(x) \sim \Sigma_\alpha s_\alpha x^\alpha \in \mathbb{C}[[T_pM]] \otimes \mathrm{End}(S_q)$$

and s_α is determined by $s_\alpha(0)$. So we need to expand the domain and range of σ.

Definition 9.24. The *Getzler symbol* is

$$\sigma : C^\infty(S \boxtimes S^*) \to C^\infty(\mathbb{C}[[T_pM]] \otimes \Lambda TM \otimes \mathrm{End}_{Cl}(S)).$$

Remark 9.25. $\mathcal{D}(S) \subseteq C^\infty(S \boxtimes S^*)$ as, for example, a differential operator is

$$(Pf)(x) = \int_M P\delta(x,y)f(y)\,\mathrm{dvol}(y)$$

i.e. pick out the diagonal with a δ-function, and $\mathcal{P}(TM) \subseteq \mathbb{C}[[TM]]$ so we have a genuine expansion.

Now if we define σ the natural way it is not an algebra homomorphism with respect to algebraic multiplication in $C^\infty(S \boxtimes S^*)$ given by convolution. However...

Proposition 9.26. If $T \in \mathcal{D}(S)$ is of order $\leq m$ and $Q \in C^\infty(S \boxtimes S^*)$ is of degree $\leq m'$ the

$$\sigma_m(T)\sigma_{m'} = \sigma_{m+m'}(TQ).$$

So we get a well defined symbol on our required elements.

Now apply to $k_t(p,q)$:

$$k_t(p,q) \sim \frac{1}{(4\pi t)^{n/2}} e^{-\frac{d(p,q)^2}{4t}} \Sigma_j t^j \Theta_j(p,q).$$

Proposition 9.27. $\Theta_j(p,q)$ has Getzler degree at most $2j$. The heat symbol

$$W = h_t(\sigma_0(\Theta_0) + t\sigma_2(\Theta_1) + ...),$$

where $h_t = \frac{1}{(4\pi t)^{n/2}} e^{-\frac{d(p,q)^2}{4t}}$, satisfies

$$\frac{\partial W}{\partial t} + \sigma_2(D^2)W = 0$$

Remark 9.28. Compare this with

$$\frac{\partial k_t(p,q)}{\partial t} + D_p^2 k_t(p,q) = 0.$$

Proof. (Sketch) Let $s = \Sigma_j \Theta_j(p,q)t^j$, then

$$\frac{\partial}{\partial t}(hs) + D^2(hs) = 0.$$

Now expand and manipulate. $s = \Sigma t^j u_j$ where $u_j \sim \Sigma u_\alpha x^\alpha$ where the u_α are synchronous. Solve order by order in t:

$$\nabla_{\frac{\partial}{\partial r}}(r^j g^{1/4} u_j) = -r^{j-1} g^{1/4} D^2 u_{j-1} \qquad (u_{-1} = 0)$$

Giving $\deg u_j \leq 2j$. Now

$$\frac{\partial}{\partial r}(r^j \sigma_{2j}(u_j)) = -r^{j-1}\sigma_2(D^2)\sigma_{2j-2}(u_{j-1}),$$

hence W has the required property. ∎

This allows us to solve for W:

$$\frac{\partial W}{\partial t} + (-\Sigma_i(\frac{\partial}{\partial x_i} + \frac{1}{4}\Sigma_j R_{ij} X^j)^2 + F^s)W = 0,$$

using 'maths magic' (chapter 8)

$$W = \frac{1}{(4\pi t)^{n/2}}\exp(-tF)\det{}^{1/2}\left(\frac{tR/2}{\sinh(tR/2)}\right)\exp\left(-\frac{1}{4t}\langle\frac{tR}{2}\coth\left(\frac{tR}{2}\right)x, x\rangle\right).$$

Recall $W = h_t \Sigma t^j \sigma_{2j}(\Theta_j)$. At $t = 1$ and $x = 0$ we have constant part:

$$\begin{aligned}\sigma^0 &= \Sigma\sigma^0_{2j}(\Theta_j)\\ &= \frac{1}{(4\pi)^{n/2}}\exp(-F)\det{}^{1/2}\left(\frac{R/2}{\sinh(R/2)}\right)\end{aligned}$$

Let

$$\begin{aligned}\mathrm{ch}(s/\Delta) &= \mathrm{tr}^{s/\Delta}\left(\exp\left(\frac{i}{2\pi}F^s\right)\right)\\ \hat{A}(TM) &= \det{}^{1/2}\left(\frac{R/2}{\sinh(R/2)}\right)\end{aligned}$$

Theorem 9.29. (Atiyah-Singer) Let M^{2n} be compact and oriented. Let $S \to M$ be the canonically graded Clifford bundle with Dirac operator D. Then

$$\mathrm{Ind}(D) = \int_M \hat{A}(TM) \wedge \mathrm{ch}(s/\Delta).$$

Proof.

$$\begin{aligned}\mathrm{Ind}(D) &= \frac{1}{(4\pi)^{n/2}}\int_M s\,\mathrm{tr}(\Theta_{n/2})\,\mathrm{dvol}\\ &= \frac{1}{(4\pi)^{n/2}}\int_M (-2i)^{n/2}\,\mathrm{tr}^{s/\Delta}(\sigma^0_n(\Theta_{n/2}))\end{aligned}$$

where $\sigma^0_n(\Theta_{n/2})$ is the n-form component of $\sigma^0(\Theta_{n/2})$. But simplifying the integrand

$$\begin{aligned}&\left(\frac{(-2i)^{n/2}}{(4\pi)^{n/2}}\det{}^{1/2}\left(\frac{R/2}{\sinh(R/2)}\right)\wedge\mathrm{tr}^{s/\Delta}\exp(-F)\right)_{n\text{-form part}}\\ =\ &\left(\det{}^{1/2}\left(\frac{iR/4\pi}{\sinh(iR/4\pi)}\right)\wedge\mathrm{tr}^{s/\Delta}\exp\left(\frac{i}{2\pi}F\right)\right)_n\\ =\ &\hat{A}(TM)\wedge\mathrm{ch}(s/\Delta)\end{aligned}$$

∎

Example 9.30. Let Δ be the spinor bundle and E be a vector bundle (to twist with), then if

$$S = \Delta \otimes E$$

we have

$$D(s \otimes e) = Ds \otimes e + \Sigma c(e_i) s \otimes \nabla_i e$$

and hence

$$\mathrm{Ind}(D) = \int_M \hat{A}(TM) \wedge \mathrm{ch}(E).$$