LECTURE 1: SURGERY, HANDLES & MORSE THEORY — 8/9/11

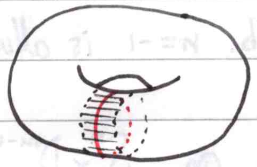
In this lecture I will explain, with plenty of examples, the interplay between surgery, handles and Morse theory.

I will touch on the homotopical effect of surgery; but this will be addressed again in the next lecture.

§1: Handles and Surgery:

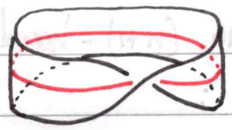
Def^n: An embedding of a submanifold $N^n \hookrightarrow M^m$ is framed if it extends to an embedding $N^n \times D^{m-n} \hookrightarrow M^m$, in which case $N^n \times \{0\} \hookrightarrow M^m$ is called the core of the framed embedding.

Examples: i)



is a framed embedding $S^1 \times D^1 \hookrightarrow S^1 \times S^1$

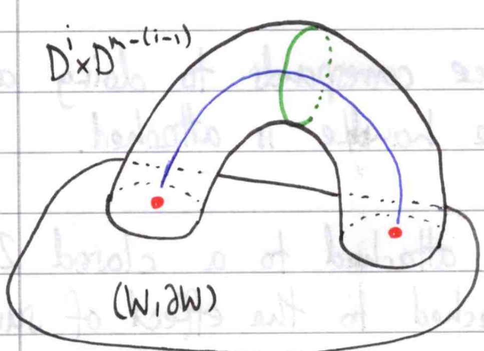
ii)



is an embedding $S^1 \hookrightarrow$ Möbius that cannot be framed.

Def^n: Given an $(m+1)$-dimensional manifold with boundary $(W, \partial W)$ and a framed embedding $S^{i-1} \times D^{m-(i-1)} \hookrightarrow \partial W$ $(0 \leq i \leq m+1)$ define the $(m+1)$-dimensional manifold with boundary $(W', \partial W')$ obtained from W by attaching an i-handle to be

$$W' := W \cup_{S^{i-1} \times D^{m-i+1}} D^i \times D^{m-(i-1)}$$



$S^{i-1} \times \{0\}$ is called the attaching sphere.

$D^i \times \{0\}$ is called the core (of the handle).

$\{0\} \times S^{m-i}$ is called the belt sphere.

Remark: Homotopically, attaching an i-handle is the same as attaching a (thickened) i-cell.

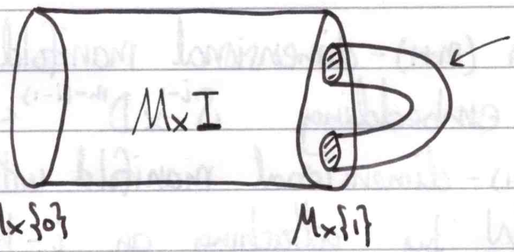
Examples: • Attaching a 0-handle is disjoint union with $D^0 \times D^{m+1}$
• Attaching an $(m+1)$-handle is filling in an $S^m \hookrightarrow \partial W$ with a D^{m+1}.

Def^n: An n-surgery on an m-dimensional manifold M^m is the surgery removing a framed embedding $g: S^n \times D^{m-n} \hookrightarrow M^m$ and replacing it with $D^{n+1} \times S^{m-n-1}$. We call

$$M' := \overline{M \setminus g(S^n \times D^{m-n}) \cup_{S^n \times S^{m-n-1}} D^{n+1} \times S^{m-n-1}}$$

the effect of the surgery. (Nb. $n=-1$ is allowed)

Def^n: The trace of the n-surgery on $S^n \times D^{m-n} \subset M^m$ is the cobordism (W, M, M') obtained by attaching an $(n+1)$-handle $D^{n+1} \times D^{m-n}$ to $M \times I$ at $S^n \times D^{m-n} \times \{1\} \subset M \times I$:

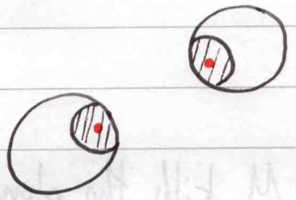


or equivalently the trace is obtained by attaching an $(m-n)$-handle to $M' \times I$.

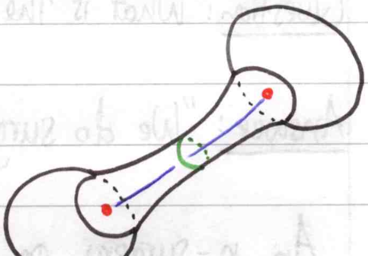
Remark: Attaching $(i+1)$-handles to the trace corresponds to doing an i-surgery on the boundary component to which the handle is attached.

Example: We show how a 1-handle attached to a closed 2-manifold can be viewed as a 2-handle attached to the effect of surgery.
Let $M = S^2 \sqcup S^2$, let's do a 0-surgery on $S^0 \times D^2 \hookrightarrow S^2 \sqcup S^2$

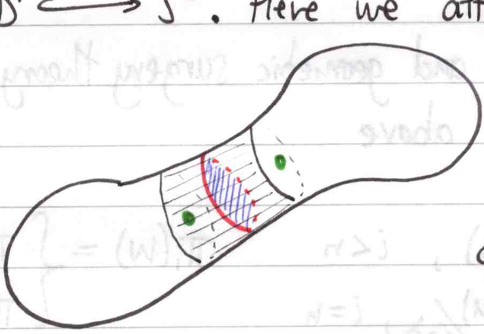
where $S^0 \times D^2$ is embedded as follows:



Attaching a 1-handle we obtain:



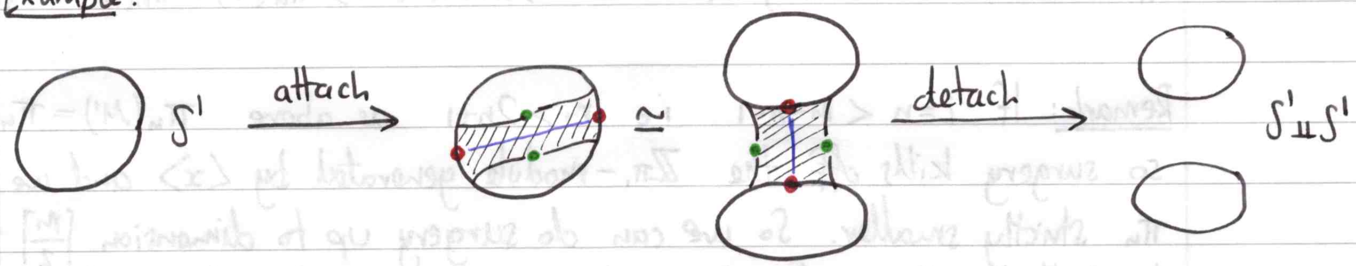
The effect of the surgery is S^2. We can view this the other way round by starting with S^2 and doing a 1 surgery on the framed belt sphere $S^1 \times D^1 \hookrightarrow S^2$. Here we attach a 2-handle inside S^2:



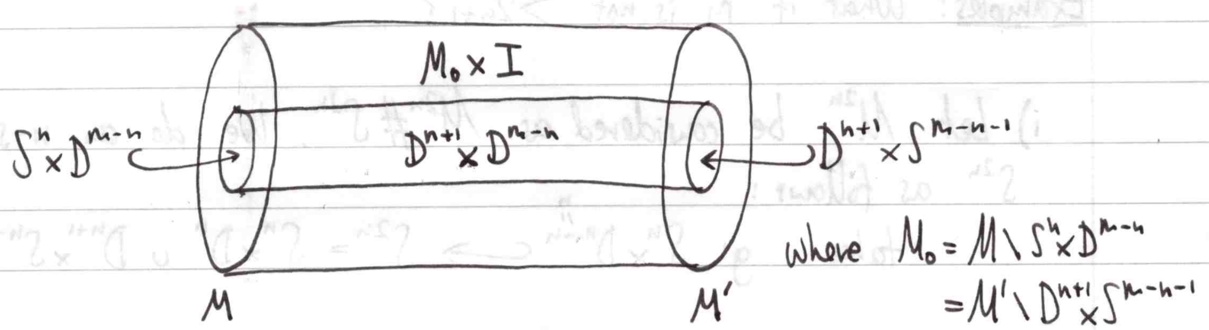
and the effect is $S^2 \amalg S^2$.

<u>Remark:</u> Doing an n-surgery is attaching a thickened $(n+1)$-cell (the core of the attaching sphere thickened) and then detaching a thickened $(m-n)$-cell (the core of the belt sphere thickened):

<u>Example:</u>



A symmetric way to draw this is (as in Andrew's book):



where $M_0 = M \setminus S^n \times D^{m-n}$
$= M' \setminus D^{n+1} \times S^{m-n-1}$

Question: What is the effect of surgery on homotopy / homology?

Answer: "We do surgery to kill"!

An n-surgery on M removing $g: S^n \times D^{m-n} \hookrightarrow M$ kills the element $x = g|: S^n \times \{0\} \hookrightarrow M$ represented by the core in $\pi_n(M)$.
The dual $(m-n-1)$-surgery removing $g': D^{n+1} \times S^{m-n-1} \hookrightarrow M'$ kills $x' = g'|: \{0\} \times S^{m-n-1} \hookrightarrow M'$ in $\pi_{m-n-1}(M')$.

Prop^n: (Ranicki - Algebraic and geometric surgery theory proposition 4.19):
For an n-surgery as above

$$\pi_i(W) = \begin{cases} \pi_i(M), & i<n \\ \pi_n(M)/\langle x\rangle, & i=n \end{cases} \qquad \pi_i(W) = \begin{cases} \pi_i(M'), & i<m-n-1 \\ \pi_{m-n-1}(M')/\langle x'\rangle, & i=m-n-1. \end{cases}$$

(See Ranicki 4.19 for details about the effect on homology, which we will not consider in this lecture.)

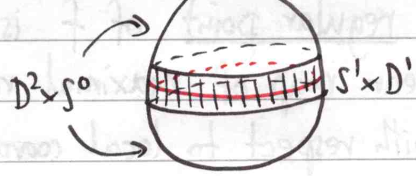
If $2n+1 \leq m$ and $i<n$, then $m-n-1 \geq n > i \Rightarrow \pi_i(M') = \pi_i(W) = \pi_i(M)$
If $2n+2 \leq m$ and $i=n$, then $m-n-1 \geq n+1 > n = i \Rightarrow \pi_n(M') = \pi_n(W) = \pi_n(M)/\langle x\rangle$

Remark: If $i = n < m-n-1$ i.e $m > 2n+1$ as above $\pi_n(M') = \pi_n(M)/\langle x\rangle$ so surgery kills off the $\mathbb{Z}\pi_1$-module generated by $\langle x\rangle$ and we make π_n strictly smaller. So we can do surgery up to dimension $\left[\frac{m}{2}\right]-1$ to kill everything. (Provided we have a supply of framed embeddings.)

Examples: What if m is not $>2n+1$?

i) Let M^{2n} be considered as $M^{2n} \# S^{2n}$. We do an n-surgery on S^{2n} as follows:
take $g: S^n \times D^{\overset{n}{\overset{\shortparallel}{m-n}}} \hookrightarrow S^{2n} = S^n \times D^n \cup D^{n+1} \times S^{n-1}$

e.g. $S^2 = S^1\times D^1 \cup D^2\times S^0$:



The effect of surgery is $M' = M \# (D^{n+1}\times S^{n-1} \cup D^{n+1}\times S^{n-1})$
$= M\#(S^{n+1}\times S^{n-1})$

$\pi_{n-1}(M\# S^{n+1}\times S^{n-1}) = \pi_{n-1}(M)\oplus\mathbb{Z}$, so an n-surgery on a $2n$-fld can change π_{n-1}.

ii) We can make $\pi_n(M^{2n})$ smaller with an n-surgery without affecting π_{n-1}:
$$g: S^n\times D^n \hookrightarrow S^n\times D^n \cup S^n\times D^n = S^n\times S^n =: M$$

then $M' = D^{n+1}\times S^{n-1}\cup S^n\times D^n = S^{2n}$ and $\pi_{n-1}(S^{2n}) = \pi_{n-1}(S^n\times S^n) = 0$.

§2: Morse Theory

We use morse theory to prove that an m-dimensional manifold M^m can be obtained from $\emptyset$ by successively attaching handles of increasing index i, giving M the structure of a CW complex.

Idea: There is a connection between Morse theory, handles and surgery: If $a<b\in\mathbb{R}$ are regular values of a Morse function $f: M^m\to\mathbb{R}$ then
$$f^{-1}([a,b]; \{a\}, \{b\})$$
is a cobordism of $(m-1)$-manifolds. If all points in $[a,b]$ are regular and $f^{-1}([a,b])$ is compact then the cobordism is a product and in particular $f^{-1}(a)\cong f^{-1}(b)$. For each non-degenerate critical point in $f^{-1}([a,b])$ we get a handle.

Let $f: M^m\to\mathbb{R}$ be a C^∞ map. The following definitions work more generally, but we write them only for maps to $\mathbb{R}$.

$\underline{\text{Def}^n}$: A <u>regular point</u> of f is an $x \in M^m$ where $df(x) : \mathbb{R}^n \to \mathbb{R}$ is a linear map of maximal rank (ie rank 1). To be maximal rank, with respect to local coordinates $(x_1, \dots, x_m)$, we must have

$$\frac{\partial f}{\partial x_i} \neq 0 \quad \text{for some } i.$$

A <u>critical point</u> of f is an $x \in M^m$ where $df(x) : \mathbb{R}^n \to \mathbb{R}$ doesn't have maximal rank, ie has rank 0. With respect to local coordinates a critical point must have $\frac{\partial f}{\partial x_i} = 0 \ \forall i$.

A <u>regular value</u> of f is a $y \in \mathbb{R}$ all of whose preimages are regular points.

A <u>critical value</u> of f is a $y \in \mathbb{R}$ with at least one critical point in its preimage.

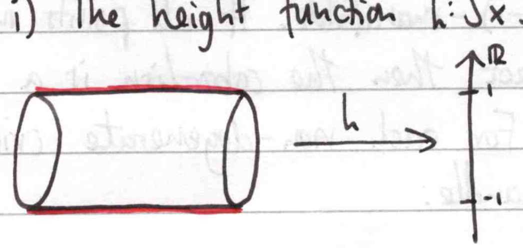
$\underline{\text{Def}^n}$: i) We say that a critical point $x \in M$ of f is <u>non-degenerate</u> if, wrt local coordinates, the Hessian matrix

$$H(x) = \left(\frac{\partial^2 f}{\partial x_i \partial x_j} \right) \text{ is invertible.}$$

ii) The <u>index</u> $\text{Ind}(x)$ of a non-degenerate critical point is the dimension of the largest negative eigenspace of $H(x)$.

iii) f is called a <u>Morse function</u> if all its critical points are non-degenerate.

<u>Examples</u>: i) The height function $h : S^1 \times I \longrightarrow \mathbb{R}$



has 2 critical lines (marked in red). We will see later that non-degen critical points must be isolated, so all critical points in the lines have to be degenerate. Alternatively we can see this from the local expression for h near the critical points:

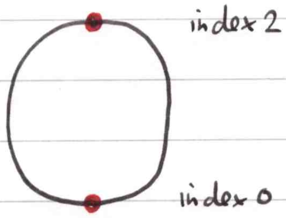
Along the top critical line $\quad h(x,y) = \sqrt{1-x^2} \quad$ (x is up)

$$\Rightarrow \frac{\partial^2 h}{\partial x^2} = -(1-x^2)^{-3/2}$$

so $\quad H(1,y) = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$ not invertible $\Rightarrow$ degenerate

Similarly for the bottom critical line.

ii) S^2 with the height function is Morse



At the north pole: $h = \sqrt{1-(x^2+y^2)}$

$\Rightarrow H(0,0) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ invertible, index 2.

Similarly get $H(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ at South pole.

Morse's lemma: Let $p \in M^m$ be non-degenerate for $f: M^m \to \mathbb{R}$. Then there exist local coordinates x_i in a neighbourhood $U \ni p$ with $x_i(p) = 0 \ \forall i$ such that

$$f = f(p) - x_1^2 - \dots - x_i^2 + x_{i+1}^2 + \dots + x_m^2$$

where i is the Index of f at p.

Proof: See 2.2 in milnmors.pdf.

Corollary: A Morse function must have isolated critical points.

Implicit function theorem: The inverse image of a regular value $y \in \mathbb{R}$ of a differentiable map $f: M^m \to \mathbb{R}$ is a submanifold $P = f^{-1}(y) \subseteq M$ with $\dim(P) = n-1$ provided $m \geq 1$.

Proof: Bredon chapter II.1

Def^n: Let $M^a = f^{-1}(-\infty, a]$ denote the ascending cobordism of the Morse function f.

$\underline{\text{Th}^m}$: Let $f: M^n \to \mathbb{R}$ be smooth. Let $a<b \in \mathbb{R}$ and $f^{-1}[a,b]$ be compact with no critical points of f. Then $M^a \cong M^b$ and M^b deformation retracts onto M^a.

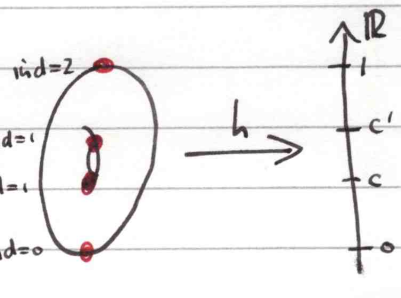
Proof: The negative gradient flow of ∇f yields the deformation retract and hence the diffeomorphism Δ.

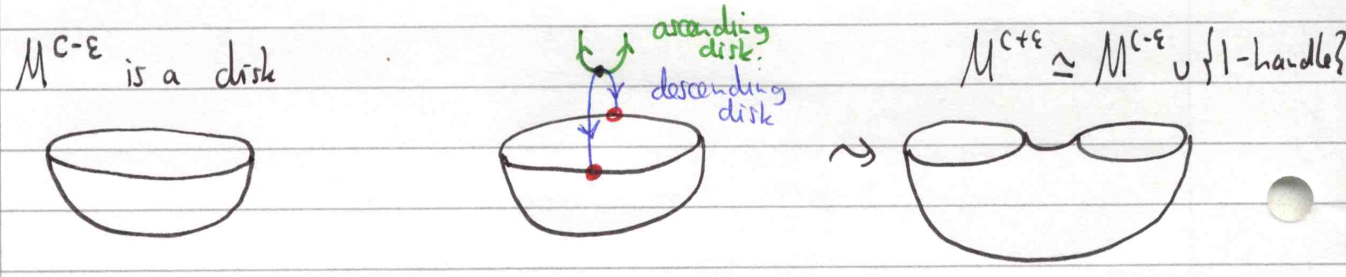
Remark: Compactness of $f^{-1}[a,b]$ is crucial; omitting just one point of $f^{-1}[a,b]$ ruins everything!

$\underline{\text{Th}^m}$: Let $f: M^n \to \mathbb{R}$ be smooth, let p be a non-degenerate critical point with index k. Setting $c=f(p)$, suppose $f^{-1}[c-\varepsilon, c+\varepsilon]$ is compact and has no critical points except p. Then $M^{c+\varepsilon}$ has the homotopy type of $M^{c-\varepsilon}$ with a k-cell attached.

Proof: (Sketch) Look at the trajectories out of p in local coordinates. The coordinates $x_1, \dots, x_k$ go 'down' with respect to f so these directions get attached to $M^{c-\varepsilon}$ via the boundary $\partial(D^k) = S^{k-1}$. "descending disk".
The coordinates $x_{k+1}, \dots, x_n$ go 'up' with respect to f so they don't get attached to $M^{c-\varepsilon}$. "ascending disk". Thus we attached a k-handle.

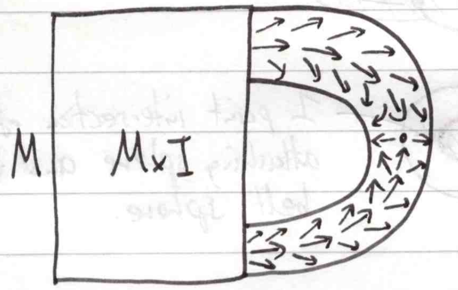
Example: $S^1 \times S^1$ with height function





Prop^n: The trace cobordism of a i-surgery admits a morse function $(W, M, M') \longrightarrow ([0,1], \{0\}, \{1\})$ with a single critical value of index $i+1$.

Proof: (by picture) The trace is $M \times I \cup \{h^{i+1}\}$ ← $(i+1)$-handle. Set $f(M) = 0$ and $f(M') = 1$, have df look like this:



ie towards $\{0\} \times \{0\} \subset D^{i+1} \times D^{n-i-1}$ in the core of the attaching sphere & away from $\{0\} \times \{0\}$ in the core of the belt sphere.

"QED"

Th^m: Let $f: M^m \to \mathbb{R}$ be a morse function s.t. M^a is compact $\forall a$, then M has the homotopy type of a CW-complex, with a cell of dim k (in fact a k-handle) for each critical point of index k.

Question: Do Morse functions always exist?
Answer: Yes! In fact they are open & dense in all functions.

Theorem: (Morse) Every m-manifold M^m admits a Morse function $f: M \to \mathbb{R}$.

Proof: M^m embeds into S^{2m+1} by Whitney's embedding theorem. Give M the subspace metric and norm, then

$$L_x : M^m \longrightarrow \mathbb{R} \qquad \text{is Morse for almost all } x \in M.$$
$$y \longmapsto \|x-y\|^2$$

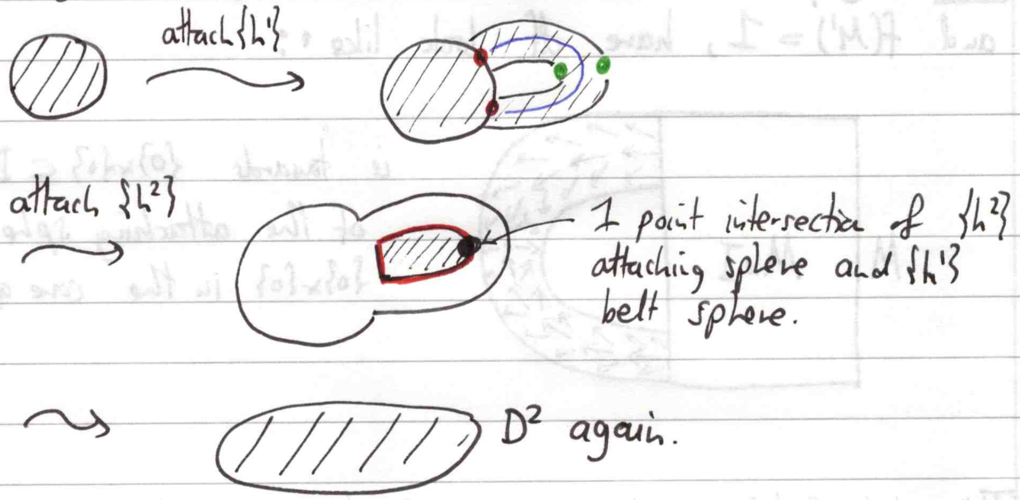
□

Thus given a cobordism (W, M, M') we may find a Morse function $f: (W, M, M') \longrightarrow ([a,b], \{a\}, \{b\})$ and from this obtain a handle decomposition for (W, M, M').

Remark: The handle decomposition is (highly) non-unique!

Two handles h^{i+1} & h^i can cancel if the attaching sphere of h^{i+1} intersects the belt sphere of h^i in a single point.

Example: D^2



Via handle cancellation we can trade an i-handle for an $(i+2)$-handle and vice versa.

$\{h^i\} \longrightarrow$ introduce $\{h^{i+1}\}$ and $\{h^{i+2}\}$ s.t they cancel **AND** s.t $\{h^i\}$ and $\{h^{i+1}\}$ cancel

$\longrightarrow \cancel{\{h^i\}} \cup \cancel{\{h^{i+1}\}} \cup \{h^{i+2}\} = \{h^{i+2}\}.$

For a more detailed explanation see my notes on the h-cobordism theorem (on the website).

Th^m: Every $(m+1)$-dimensional cobordism (W, M, M') admits a handle decomposition

$$W = (M \times I) \cup \bigcup_{j=0}^{k} h^{i_j+1} \qquad -1 \leq i_0 \leq \ldots \leq i_k \leq n$$

i.e where handles are attached in ascending order of k in $D^k \times D^{n-k}$.

Pf: (Sketch) Either we note that we can always slide an $\{h^i\}$ off an $\{h^j\}$ for $i<j$ (c.f my hcobnotes) or we appeal to the existence of

self-indexing Morse functions, ie $f: M^n \to \mathbb{R}$ s.t $\forall x \in M^n$ critical $f(x) = \mathrm{Ind}(x)$. The handle decomposition obtained from a self-indexing Morse function is one where we attach all 1-handles together, then all 2-handles, ... etc. $\square$.

<u>Remark:</u> Handle decompositions correspond to Morse functions; if we replace f with $-f$, we "turn the handle decomposition upside down". An i-handle becomes an $(n-i)$-handle

<u>justification:</u> $-f$ has the same critical points and all negative and positive eigenspaces are interchanged. $\square$.

<u>NEXT TIME:</u> local coefficients, the handle chain complex and Poincaré duality.