LECTURE 2: POINCARÉ DUALITY, LOCAL COEFFICIENTS & THE HANDLE CHAIN COMPLEX
12/9/11

The algebraic effect of a geometric surgery on an m-dimensional manifold M is determined by the Poincaré duality isomorphisms $H^{m-*}(M) \xrightarrow{\cong} H_*(M)$. This is a global expression of the local manifold property that $\forall x \in M$, $H^{m-*}(\{x\}) \xrightarrow{\cong} H_*(M, M \setminus \{x\})$. To be able to piece these local isomorphisms together we require orientability!

§1: Orientations and Poincaré duality:

Def^n: A local orientation of a manifold M at a point x is a choice of generator $\mu_x \in H_m(M, M \setminus \{x\}; \mathbb{Z}) \cong H_m(\mathbb{R}^m, \mathbb{R}^m \setminus \{0\}; \mathbb{Z}) = \mathbb{Z}$

A (global) orientation of a manifold M is a consistent choice of local orientations at all points $x \in M$. i.e. a function $\mu : x \longmapsto \mu_x$ satisfying a 'local consistency' condition: $\forall x \in M$ $\exists$ open ball $B \ni x$ of finite radius s.t. $\exists$ μ_B generating $H_m(M, M-B)$ mapping to μ_y $\forall y \in B$ under the map $H_m(M, M-B) \longrightarrow H_m(M, M-\{y\})$.

We say M is orientable if an orientation exists.

Every manifold M has an orientable two-sheeted covering $M_{\mathbb{Z}_2} \to M$:
$$M_{\mathbb{Z}_2} = \{\mu_x \mid x \in M,\ \mu_x \text{ a local orientation of } M \text{ at } x\}$$
Clearly over each point x there are the two points corresponding to $\{\pm 1\} \in \mathbb{Z}$. We can topologise $M_{\mathbb{Z}_2}$ to make it a covering space. Given a ball $B \ni x$ as above and μ_B generating $H_m(M, M-B)$ we have a corresponding open set in the topology for $M_{\mathbb{Z}_2}$:
$$U(\mu_B) = \{\mu_x \in M_{\mathbb{Z}_2} \mid x \in B \ \&\ H_m(M, M-B) \to H_m(M, M-\{x\}) \text{ sends } [\mu_B] \mapsto [\mu_x]\}$$
These open sets $U(\mu_B)$ are 'little open balls' on a single sheet of the cover. It is easy to see $M_{\mathbb{Z}_2}$ is orientable.

Def^n: If M is connected we define the orientation character or the 1^{st} Stiefel-Whitney class $w : \pi_1 M \longrightarrow \{\pm 1\} = Aut(\mathbb{Z})$ by $w(\gamma) = \begin{cases} 1, & \gamma \text{ lifts to a loop in } M_{\mathbb{Z}_2} \\ -1, & \gamma \text{ lifts to a path in } M_{\mathbb{Z}_2} \end{cases}$

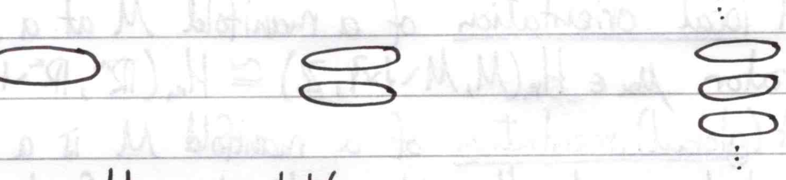
The first Stiefel-Whitney class is the obstruction to being able to orient M.

$M_{\mathbb{Z}_2} \to M$ can be embedded into a larger covering space $M_{\mathbb{Z}} \to M$, called the orientation sheaf of M, where

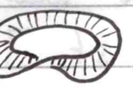
$$M_{\mathbb{Z}} = \{\alpha_x \mid x \in M,\ \alpha_x \in H_m(M, M-\{x\})\}. \quad \text{(not just generators now)}$$

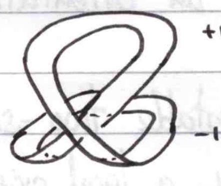
We can topologise $M_{\mathbb{Z}}$ exactly as we did for $M_{\mathbb{Z}_2}$. We get a copy of M at $\alpha_x \equiv 0$, and a copy of $M_{\mathbb{Z}_2}$ for each positive integer k given by $\pm k\mu_x$ for μ_x a generator of $H_m(M, M-\{x\})$.

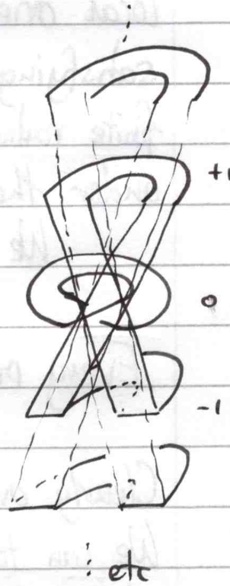
<u>Examples</u>: i) $M = S^1$ $\qquad M_{\mathbb{Z}_2} = S^1 \times \mathbb{Z}_2$ $\qquad M_{\mathbb{Z}} = S^1 \times \mathbb{Z}$



since M is orientable.

ii) Let M be the Möbius band: 

$M_{\mathbb{Z}_2} =$ 

$M_{\mathbb{Z}} =$ 

<u>Def</u>: A continuous map $M \to M_{\mathbb{Z}}$, $x \mapsto \alpha_x \in H_m(M, M-\{x\})$ is called a <u>section</u> of the covering space.

<u>Note</u> that an orientation of M <u>is</u> a section $x \mapsto \mu_x$ s.t. μ_x is a generator of $H_m(M, M-\{x\})$ for all x.

The zero section does not give an orientation as 0 is not a generator of $\mathbb{Z}$!

Now replace $\mathbb{Z}$ with any commutative ring R with identity.

<u>Def^n</u>: An <u>R-orientation</u> is a section $M \to M_R$ s.t μ_x is a generator of $H_n(M, M-\{x\}; R) = R$ for all x, i.e μ_x is a unit $\in R^\times$, and we have local consistency.

The fact that $H_n(M, M-\{x\}; R) \cong H_n(M, M-\{x\}; \mathbb{Z}) \otimes_{\mathbb{Z}} R$ let's us deduce the following: $M_R = \bigcup_{r \in R} M_r$ where $M_r = \{\pm \mu_x \otimes r \mid \underset{\mathbb{Z}\text{ coeffs}}{\mu_x \text{ generator}}\}$

$$M_r = M \text{ if } r = -r \quad (\text{e.g. if } r \text{ has order } 2)$$
$$M_r = M_{\mathbb{Z}_2} \text{ otherwise}$$

<u>Corollary</u>: i) M is $\mathbb{Z}$-orientable $\Rightarrow$ M is R-orientable $\forall R$ (since $1 \in R$).
ii) If M is not $\mathbb{Z}$-orientable then if R has an element of order 2 $\exists M_r = M$ so $\exists$ R-orientation so M is R-orientable.
iii) Every manifold is $\mathbb{Z}_2$ orientable ($1 \in \mathbb{Z}_2$ has order 2).

This tells us that the most important rings to consider are $\mathbb{Z}$ and $\mathbb{Z}_2$.

<u>Theorem</u>: If M is closed and R-orientable, then $H_n(M; R) \longrightarrow H_n(M, M-\{x\}; R) \cong R$ is an isomorphism $\forall x \in M$.

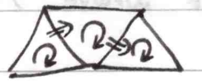
<u>Remark</u>: We need M to be closed for the R-orientation $M \to M_R$ to yield a class in $H_n(M; R)$ as cycles are closed!

<u>Def^n</u>: An element $[M] \in H_n(M; R)$ mapping to a generator $\mu_x \in H_n(M, M-\{x\}; R)$ $\forall x$ is called a <u>fundamental class</u> for M with R coefficients.

By the theorem above if M is closed and R-orientable then there exists a fundamental class.

Suppose our manifold M has the structure of a simplicial complex, then the fundamental class $[M]$ is a linear combination of m-simplices $\sum_i k_i \sigma_i$. For $[M]$ to map to a generator of $H_m(M, M-\{x\}; \mathbb{Z})$ $\forall x \in \mathring{\sigma}_i$ we must have $k_i = \pm 1$. If we have determined a sign for k_i, since $[M]$ is a cycle this determines the value of k_j for all adjacent m-simplices σ_j (ie simplices that share a $(m-1)$-face).

e.g



etc.

A choice of signs k_i making $[M] = \sum_i k_i \sigma_i$ a cycle is only possible $\Leftrightarrow$ M is orientable.

Note that with $\mathbb{Z}_2$ coefficients $[M] = \sum_i \sigma_i$ is a $\mathbb{Z}_2$ fundamental class.

<u>Poincaré duality theorem</u>: For any closed R-orientable m-dimensional manifold M, if $[M] \in H_m(M; R)$ is a fundamental class then

$$[M] \frown - : H^*(M;R) \xrightarrow{\simeq} H_{m-*}(M;R)$$

is an isomorphism.

<u>Remark</u>: By the previous considerations any closed manifold M has $\mathbb{Z}_2$ Poincaré duality.

<u>§2: local coefficients</u>

- Local coefficients are a tool to organise information about the action of $\pi_1(M)$ on various abelian groups ($\rightsquigarrow$ modules, e.g $C_*(\tilde{M})$, $\pi_i(M)$ $i \geq 2$, etc.)
- Local coefficients give a fundamental class for non-orientable manifolds and thus a way to extend Poincaré duality with $\mathbb{Z}$ coefficients to non-orientable (hence non-simply connected) manifolds.
- We have already been implicitly talking about local coefficients in this lecture!

There are two equivalent points of view:

i) The fundamental chain complex of interest associated to any manifold M is that of its universal cover $\tilde{M}$, viewed as a chain complex over $\mathbb{Z}\pi_1(M)$. From this point of view local coefficients <u>are</u> modules over the fundamental group ring $\mathbb{Z}\pi_1(M)$.

ii) A more topological perspective: we consider fibre bundles over M with fibre an <u>abelian</u> group G & transition functions in $\mathrm{Aut}_{\mathbb{Z}}(G)$. (Recall the orientation sheaf $M_{\mathbb{Z}} \to M$). We get a chain complex of formal sums of singular simplices (or cells) with coefficients in the fibre over the simplex (c.f. $[M]$ for a simplicial complex.)

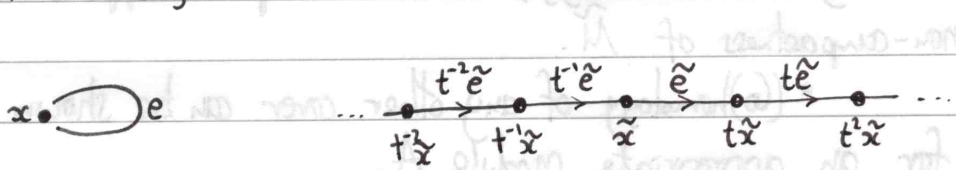
<u>Remark</u>: In general $\pi_1(M)$ may not be commutative so we must distinguish between left and right $\mathbb{Z}\pi_1$-modules. We can switch between left and right via $g \cdot m = m \cdot g^{-1}$.

First note that $C_*(\tilde{M})$ is a right $\mathbb{Z}\pi_1$-module for singular or cellular chains where $\pi_1(M)$ acts by deck transformations and we extend linearly to $\mathbb{Z}\pi_1(M)$.

e.g. <u>Singular</u>: let $\sigma : \Delta^k \to \tilde{M}$, $g \in \pi_1(M)$ then $\sigma \cdot g := \Delta^k \xrightarrow{\sigma} \tilde{M} \xrightarrow{g} \tilde{M}$

<u>Cellular</u>: $[e] \in C_*(\tilde{M})$, then $[e] \cdot g := [g(e)]$ the g-translate of e.

<u>Example</u>: Let $M = S^1$, $\tilde{M} = \mathbb{R}$



$\pi_1(M) = \mathbb{Z}\langle t \rangle$ 

$$C_1(\mathbb{R}) = \underset{\mathbb{Z}\pi_1(M)}{\underset{\parallel}{\mathbb{Z}[t,t^{-1}]}}\langle \tilde{e} \rangle \xrightarrow{1-t} \underset{\mathbb{Z}\pi_1(M).}{\underset{\parallel}{\mathbb{Z}[t,t^{-1}]}}\langle \tilde{x} \rangle = C_0(\mathbb{R})$$

Def^n: Given a left $\mathbb{Z}\pi_1(M)$-module A, form $C_*(M;A) := C_*(\tilde{M}) \otimes_{\mathbb{Z}\pi_1} A$. The boundary map of this chain complex is $d_{C_*(\tilde{M})} \otimes 1$. The homology of this chain complex is called the <u>homology of M with local coefficients in A</u>.

In the case that $A = A_\rho$, the $\mathbb{Z}\pi_1(M)$-module corresponding to the representation $\rho: \pi_1(M) \longrightarrow \mathrm{Aut}_{\mathbb{Z}}(A)$ it is common to call $H_*(M;A_\rho)$ the <u>homology of M twisted by ρ</u>.

Def^n: Given a left $\mathbb{Z}\pi_1(M)$-module A, consider $C_*(\tilde{M})$ as a left module via $g \cdot m := m \cdot g^{-1}$ and form the cochain complex $C^*(M;A) := \mathrm{Hom}_{\mathbb{Z}\pi_1}(C_*(\tilde{X}), A)$. $H^*(M;A)$ is called the <u>cohomology of M with local coefficients in A</u> / <u>twisted by ρ</u> (for $H^*(M;A_\rho)$).

<u>Examples:</u> • Let $\rho: \pi_1(M) \longrightarrow \mathrm{Aut}(A)$ be the trivial homomorphism, A_ρ the associated module, then $C_*(\tilde{M}) \otimes_{\mathbb{Z}\pi_1} A_\rho \cong C_*(M) \otimes_{\mathbb{Z}} A$ since $\pi_1(M)$ acting trivial on A_ρ quotients at its action on $C_*(\tilde{M})$ to just $C_*(M)$. Taking homology we get $H_*(M;A_\rho) = H_*(M;A)$ the usual homology with coefficients in A. In particular if $A = \mathbb{Z}$ this is just usual homology, (untwisted!). Similarly for cohomology.

• For $A = \mathbb{Z}\pi_1$ with the standard action, $C_*(\tilde{M}) \otimes_{\mathbb{Z}\pi_1} \mathbb{Z}\pi_1 \cong C_*(\tilde{M})$ $\Rightarrow H_*(M;\mathbb{Z}\pi_1) = H_*(\tilde{M})$ the untwisted homology of the universal cover. Similarly for cohomology, but with compact supports due to the possible non-compactness of $\tilde{M}$.

• (Co)homology of any other cover can be shown to be $H_*(M;A)$ for an appropriate module A.

<u>Example/exercise:</u> Let $\mathbb{Z}w$ be the $\mathbb{Z}\pi_1$-module associated to the first Stiefel-Whitney class $w: \pi_1(M) \longrightarrow \mathrm{Aut}(\mathbb{Z})$

<u>Compute:</u> $H_*(K;\mathbb{Z}w)$ for K the Klein bottle.

<u>Hint:</u> $C_*(M;\mathbb{Z}w) = C_*(\tilde{M}) \otimes_{\mathbb{Z}\pi} \mathbb{Z}w = C_*(M_{\mathbb{Z}_2}) \otimes_{\mathbb{Z}[\mathbb{Z}_2]} \mathbb{Z}_-$
where $\mathbb{Z}_-$ is the $\mathbb{Z}[\mathbb{Z}_2]$-module with $\mathbb{Z}_2$ acting as -1.

<u>Twisted Poincaré duality</u>: Let M^m be connected, compact, $\partial M = \emptyset$, then $H_m(M;\mathbb{Z}_w) \cong \mathbb{Z}$. Let $[M] \in H_m(M;\mathbb{Z}_w)$ generate, then

$$\cap[M]: H^k(M;\mathbb{Z}_w) \longrightarrow H_{m-k}(M;\mathbb{Z})$$
$$\cap[M]: H^k(M;\mathbb{Z}) \longrightarrow H_{m-k}(M;\mathbb{Z}_w)$$

are isomorphisms.

§3: The $\mathbb{Z}\pi_1$-module handle chain complex

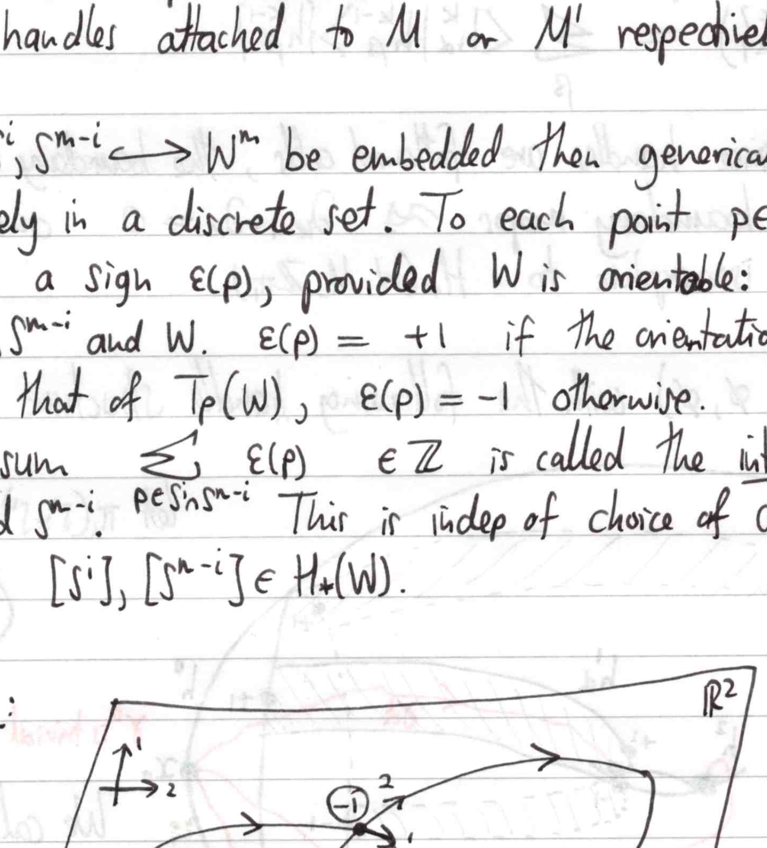
Now we have developed the right language, we return to explain the handle chain complex of a cobordism.

Let $(W^m; M, M')$ be a cobordism, then via a Morse function we obtain a handle decomposition for W. To this we can associate the $\mathbb{Z}\pi_1(W)$-module handle chain complexes $C_*(W, M; \mathbb{Z}\pi_1(W))$ & $C_*(W, M'; \mathbb{Z}\pi_1(W))$ viewing W as handles attached to M or M' respectively.

Let $S^i, S^{m-i} \hookrightarrow W^m$ be embedded then generically they intersect transversely in a discrete set. To each point $p \in S^i \cap S^{m-i}$ we can associate a sign $\varepsilon(p)$, provided W is orientable: choose orientations for S^i, S^{m-i} and W. $\varepsilon(p) = +1$ if the orientation of $T_p(S^i) \oplus T_p(S^{m-i})$ matches that of $T_p(W)$, $\varepsilon(p) = -1$ otherwise.

The sum $\sum_{p \in S^i \cap S^{m-i}} \varepsilon(p) \in \mathbb{Z}$ is called the <u>intersection number</u> $S^i \cdot S^{m-i}$ of S^i and S^{m-i}. This is indep of choice of orientations and depends only on $[S^i], [S^{m-i}] \in H_*(W)$.

<u>Example</u>:



$S^1 \cdot S^{2-1} = -1 + 1 = 0$

Algebraically the spheres do not intersect. Geometrically they can be homotoped not to intersect. c.f. Whitney trick!

$\mathbb{Z}\pi_1(W)$ - handle chain complex:

- Choose a basepoint $x_0 \in M$ (or anywhere in W).
- For each i-handle $\{h^i_\alpha\}$ choose a path γ_α from x_0 to the centre of the handle $\{0\}\times\{0\} \in D^i \times D^{n-i}$, called "threaded handles".
- For an i-handle $\{h^i_\alpha\}$ attached to an $(i-1)$-handle $\{h^{i-1}_\beta\}$ define the incidence number as

$$\langle h^i_\alpha | h^{i-1}_\beta \rangle = \sum_p \varepsilon(p) \gamma_\beta^{-1} *_p \gamma_\alpha \in \mathbb{Z}\pi_1(W)$$

where $\gamma_\beta^{-1} *_p \gamma_\alpha$ is the loop $x_0 \xrightarrow{\gamma_\alpha}$ centre of $h^i_\alpha \xrightarrow{\text{path}} p$
$\longrightarrow$ centre of $h^{i-1}_\beta \xrightarrow{\gamma_\beta^{-1}} x_0$

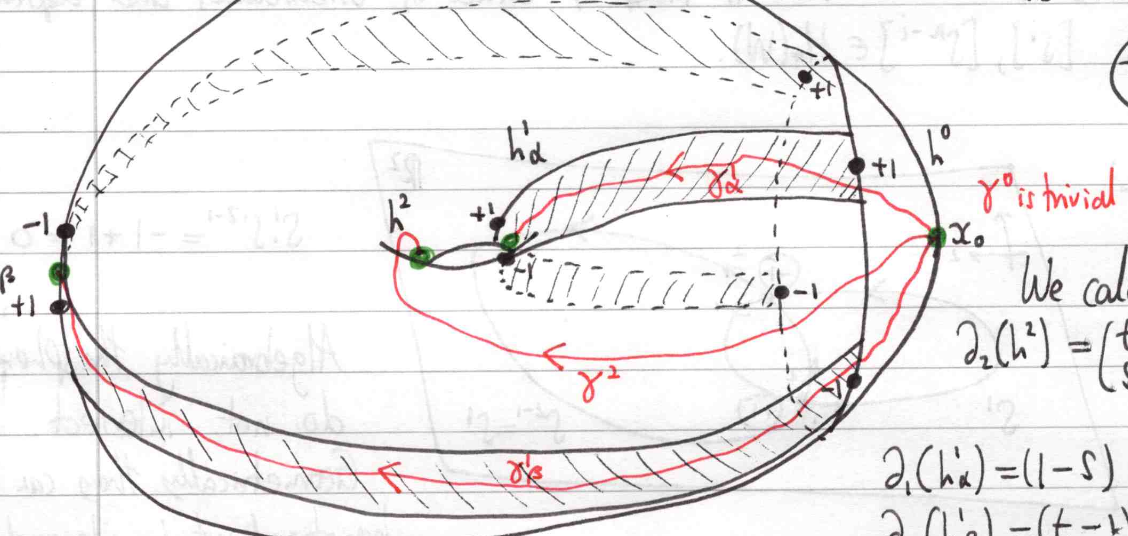
unique up to homotopy, and $\sum_p$ is a sum over all intersection points of the attaching sphere of h^i_α and the belt sphere of h^{i-1}_β.

- Define $C_k(W, M; \mathbb{Z}\pi_1) = \mathbb{Z}[\pi_1(W)]\{k\text{-handles}\}$, this is a $\mathbb{Z}[\pi_1(W)]$-module chain complex (for handles attached to M).

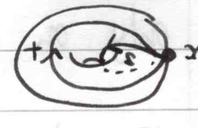
$$\partial_k(\{h^k_\alpha\}) = \sum_\beta \langle h^k_\alpha | h^{k-1}_\beta \rangle \{h^{k-1}_\beta\}$$

Note that since handles are fattened cells, the boundary maps are just like cellular boundary maps $\Rightarrow \partial_{k-1} \circ \partial_k = 0$ and the homology is naturally isomorphic to $H_*(W, M; \mathbb{Z}\pi_1)$

Example: 1) $(S^1\times S^1, \emptyset, \emptyset)$ with the following handle structure:



let $\pi_1(S^1\times S^1)$ have generators:



We calculate

$$\partial_2(h^2) = \begin{array}{cc} \begin{pmatrix} t-1 \\ s-1 \end{pmatrix} & \begin{array}{l} \leftarrow \alpha \\ \leftarrow \beta \end{array} \end{array}$$

$$\partial_1(h^1_\alpha) = (1-s)$$
$$\partial_1(h^1_\beta) = (t-1)$$

So we get the $\mathbb{Z}\pi_1(W) = \mathbb{Z}[\mathbb{Z}^2]$-module chain complex:

$$\mathbb{Z}[\mathbb{Z}\langle t\rangle \oplus \mathbb{Z}\langle s\rangle]\langle h^2\rangle \xrightarrow{\binom{t-1}{s-1}} \mathbb{Z}[\mathbb{Z}\langle t\rangle \oplus \mathbb{Z}\langle s\rangle]\langle h^1_\alpha\rangle \oplus \mathbb{Z}[\mathbb{Z}\langle t\rangle \oplus \mathbb{Z}\langle s\rangle]\langle h^1_\beta\rangle \xrightarrow{(1-s,\,t-1)} \mathbb{Z}[\mathbb{Z}\langle t\rangle \oplus \mathbb{Z}\langle s\rangle]\langle h^0\rangle$$

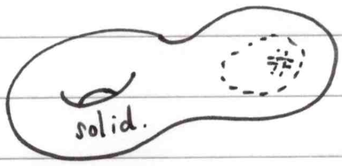
$\binom{t-1}{s-1}$ is injective since $\sum_{i,j} a_{ij} t^i s^j \xmapsto{t-1} 0 \iff a_{i,j} = a_{i-1,j}\ \forall i,j$ but only finitely many a_{ij} are non-zero $\Rightarrow$ all must be zero, so injective

Thus $\overset{H_2(S^1\times S^1;\,\mathbb{Z}\pi_1)}{\overset{\shortparallel}{H_2(W,M;\mathbb{Z}\pi_1)}} = \overset{\shortparallel}{H_1(W,M;\mathbb{Z}\pi_1)} = 0$.

$$\underset{H_0(S^1\times S^1;\,\mathbb{Z}\pi_1)}{\underset{\shortparallel}{H_0(W,M;\mathbb{Z}\pi_1)}} = \mathbb{Z}[\mathbb{Z}\langle t\rangle \oplus \mathbb{Z}\langle s\rangle]\langle h^0\rangle \Big/ {\scriptstyle t=1,\ s=1} = \mathbb{Z}\langle h^0\rangle$$

This agrees with $H_*(S^1\times S^1;\mathbb{Z}\pi_1) \cong H_*(\widetilde{S^1\times S^1};\mathbb{Z}) = H_*(\mathbb{R}^2;\mathbb{Z})$.

2) Exercise: Let M = fattened $S^1\vee S^2$ $\quad (N(S^1\vee S^2),\ \emptyset,\ S^1\times S^1 \sqcup S^2)$



Give M a handle decomposition with a 0, 1 & 2 handle and compute the $\mathbb{Z}[t,t^{-1}]$-module chain complex $C_*(M;\mathbb{Z}[t,t^{-1}])$ and its homology.

—||—

References: Davis & Kirk, Hatcher, Ranicki – Algebraic & Geometric Surgery

NEXT TIME: BUNDLES.