LECTURE 3: VECTOR BUNDLES AND SURGERY 15/9/11

Motivation/Recap

Suppose we want to kill $x \in \pi_n(M^m)$. We have:

Whitney Embedding Theorem

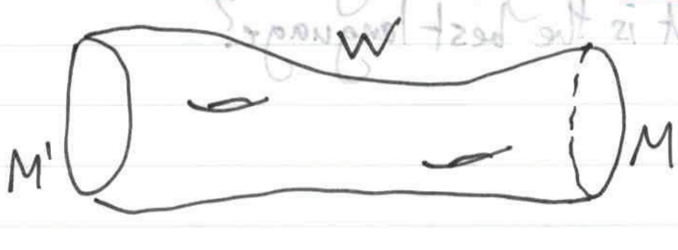
If $f: N^n \to M^m$ a map of manifolds and

either • $2n+1 \leq m$
or • $m = 2n \geq 6$ and $\pi_1(M) = 0$

then f is homotopic to an embedding. $\square$

If an embedding $f: S^n \hookrightarrow M^m$ can be extended to a framed embedding $\bar{f}: S^n \times D^{m-n} \hookrightarrow M^m$, we can do surgery on M to kill the htpy class of the embedded S^n.

Recall for M' the effect and W the trace we had



• $2n+1 \leq m \Rightarrow \pi_i(M) = \pi_i(W) = \pi_i(M') \quad \forall i < n$
• $2(n+1) \leq m \Rightarrow \pi_n(M') = \pi_n(M) / \langle x \rangle$

So we know to some extent what should happen to homotopy after surgery. However surgery affects more than this!

EXAMPLE $S^0 \hookrightarrow S^2$ with 2 different framings.

Framing 1

Framing 2 $S^0 \times D^2$

[then attach $D^1 \times S^1$ with "trivial" framing ]
□

$\Downarrow$ $\Downarrow$

Torus Klein bottle.

MORAL: We need to keep track of the framing!

e.g. • How many framings?
• What is the best language?

Outline

1. Fibrations
2. Vector Bundles
3. Normal Bundles + Surgery
4. Examples.

① FIBRATIONS

<u>Def'n</u> A fibration is a sequence of spaces

$$F \longrightarrow E \xrightarrow{p} X$$

such that it has "htpy lifting":

$$\begin{tikzcd} & E \arrow[d, "p"] \\ Y \times I \arrow[ur, dashed] \arrow[r, "h"] & X \end{tikzcd}$$

(after fixing $Y \times \{0\}$).

A fibre bundle is a sequence of spaces

$$F \longrightarrow E \longrightarrow X$$

that is "locally trivial": $\forall x \in X \ \exists U \ni x$ s.t.

$$\begin{tikzcd} \phi : p^{-1}(U) \arrow[rr] \arrow[dr, "p"'] & & U \times F \arrow[dl, "pr."] \\ & U & \end{tikzcd}$$

<u>Remark</u> A fibre bundle is a fibration

EXAMPLE

- Covering maps $\quad \mathbb{Z}_n \longrightarrow E \xrightarrow{n:1} X$
- Vector bundles $\quad \mathbb{F}^N \longrightarrow V \longrightarrow X$
- The Hopf bundle $\quad S^1 \longrightarrow S^{2n+1} \longrightarrow \mathbb{C}P^n$
- $\pi_1(SO(N)) = \mathbb{Z}_2$ for $N \geq 3$
  $$\Rightarrow \mathbb{Z}_2 \longrightarrow Spin(N) \xrightarrow[\rho]{2:1} SO(N)$$
- Let $Spin^c(N) = Spin(N) \times_{\mathbb{Z}_2} U(1)$
  $$= \{(A, \lambda)\} / _{(A,\lambda) \sim (-A, -\lambda)}$$

$$\Rightarrow \qquad U(1) \longrightarrow Spin^c(N) \longrightarrow SO(N)$$
$$(A, \lambda) \longmapsto \rho(A)$$

MORE LATER!

(2) VECTOR BUNDLES

Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ and

$$\mathbb{F}^N \longrightarrow V \xrightarrow{\pi} X$$

be a vector bundle. Triv cover $\{U_\alpha\}$

$$\begin{tikzcd} \pi^{-1}(U_\alpha) \arrow[rr, "t_\alpha"] \arrow[dr, "\pi"'] & & U_\alpha \times \mathbb{F}^N \arrow[dl, "pr"] \\ & U_\alpha & \end{tikzcd}$$

transition functions for $U_\alpha \cap U_\beta \neq 0$

$$\psi_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow GL(N, \mathbb{F}) \quad \text{cont.}$$

Remark • By picking a metric can take $\psi_{\alpha\beta}$ in max compact subgroup

$$\text{cx} \Rightarrow \psi_{\alpha\beta}(x) \in U(N)$$
$$\text{real} \Rightarrow \psi_{\alpha\beta}(x) \in O(N)$$

n.b. Gram-Schmidt at def retract.

Def'n A vector bundle is trivial if $V \cong X \times \mathbb{F}^N = \varepsilon^N$ and a choice of ~~fra~~ iso is a framing.

EXAMPLE

- Tangent bundle $\quad \mathbb{R}^m \longrightarrow TM \longrightarrow M$
- Tautological bundle

$$\mathbb{R} \longrightarrow \{(x,\ell) \in \mathbb{R}^{n+1} \times \mathbb{R}P^n \mid x \in \ell\} \longrightarrow \mathbb{R}P^n$$

$$\mathbb{C} \longrightarrow \{(x,\ell) \in \mathbb{C}^{n+1} \times \mathbb{C}P^n \mid x \in \ell\} \longrightarrow \mathbb{C}P^n$$

Def'n The Grassmanian $G_n(\mathbb{F}^{n+k})$ to be the set of all n-planes through the origin of $\mathbb{F}^{n+k}$ and

$$\mathbb{F}^n \to \tau^n = \{(x,V) \in G_n(\mathbb{F}^{n+k}) \times \mathbb{F}^{n+k} \mid x \in V\} \longrightarrow G_n(\mathbb{F}^{n+k})$$

to be the tautological bundle.

$\mathbb{F}^n \subset \mathbb{F}^{n+1} \subset \mathbb{F}^{n+2} \subset \ldots \quad \Rightarrow$ take a direct limit of Grassmanians

Def'n $\quad \displaystyle\varinjlim G_n(\mathbb{R}^{n+k}) =: BO(n)$

$$\varinjlim G_n(\mathbb{C}^{n+k}) =: BU(n)$$

and we can form tautological bundles in the same way.

Now given a map $f: X \to BO(n)$ we can pull back τ^n:

$$\begin{tikzcd} f^*\tau^n \arrow[r] \arrow[d] & \tau^n \arrow[d] \\ X \arrow[r, "f"] & BO(n) \end{tikzcd}$$

to form a real n-plane bundle over X. Astonishingly, all vector bundles over X can be constructed like this.

Theorem (Steenrod) There are bijections:

(1) $\left\{ \begin{array}{l} \text{Iso classes of real} \\ n\text{-plane bundles over } X \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Htpy classes of maps} \\ X \to BO(n) \end{array} \right\}$

(2) $\left\{ \begin{array}{l} \text{Iso classes of complex} \\ n\text{-plane bundles over } X \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Htpy classes of maps} \\ X \to BU(n) \end{array} \right\}$

Moreover, if $\tilde{G}_n(\mathbb{R}^{n+k})$ are the oriented n-planes then the same process gives

$$\varinjlim \tilde{G}_n =: BSO(n)$$

and

(3) $\left\{ \begin{array}{l} \text{Iso classes of oriented} \\ \text{real } n\text{-plane bundles} \\ \text{over } X \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Htpy classes of maps} \\ X \to BSO(n) \end{array} \right\}$

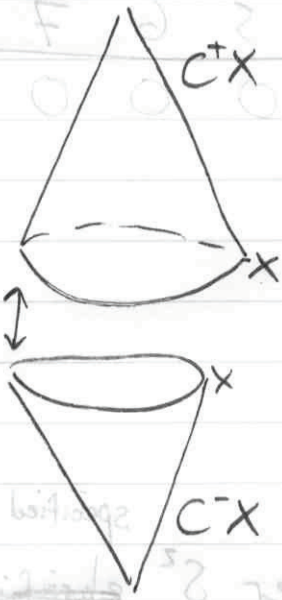
$\square$

Remark This allows us to talk interchangebly about v.b.s and their "classifying maps". We shall do so.

Clutching Functions

Given X, build v.b. E_f over SX by map

$$f : X \longrightarrow O(N) \text{ or } U(N).$$



Glue $C^{\pm}X \times \mathbb{F}^k$ by f:

$$E_f := C^-X \times \mathbb{F}^k \sqcup_\sim C^+X \times \mathbb{F}^k$$

by $(p,v) \sim (p, f(p)v)$ in n'hood of X.

<u>EXAMPLE</u> $f(z) = z$ and $X = S^1$ gives a vector bundle

$$E_f \longrightarrow X$$

whose associated sphere bundle is the Hopf bundle.

<u>Proposition</u> This construction defines a bijection

$$\left\{\begin{matrix}\text{iso class of cx} \\ \text{n-plane bundles} \\ \text{over } X\end{matrix}\right\} \longleftrightarrow [X, U(N)]$$

$$\left\{\begin{matrix}\text{iso class of} \\ \text{oriented real} \\ \text{n-plane bund over } X\end{matrix}\right\} \longleftrightarrow [X, SO(N)]$$

Corollary Taking $I \in O(N)$ as the basepoint

$$\pi_{n-1}(O(N)) = \pi_n(BO(N))$$

FACTS For N large we can calculate

	1	2	3	4	5	6	7	8
$\pi_{n-1}(O(N))$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$

EXAMPLE $n=2$, $N=2$

$$\Rightarrow \pi_1(O(2)) = \pi_1(SO(2)) = \pi_1(S^1) = \mathbb{Z}.$$

So there a $\mathbb{Z}$ 2-plane bundles over S^2 specified by htpy class of clutching functions:

$$F: S^1 \longrightarrow SO(2)$$

i.e. $$f_n(\theta) = \begin{pmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{pmatrix}.$$

represents $[f_n] = n \in \mathbb{Z} \cong \pi_1(SO(2))$

Remark As $SO(2) \simeq U(1)$ these are all underlying real bundles of complex line bundles

$$\mathcal{O}(n) \longrightarrow S^2$$

With $\mathcal{O}(1)$ the Hopf bundle!

Stable Bundles

Given two vector bundles

$$\xi : X \longrightarrow BO(j)$$
$$\zeta : X \longrightarrow BO(k)$$

with total spaces $E(\xi)$ and $E(\zeta)$, we can form a $(j+k)$-plane bundle $E(\xi) \oplus E(\zeta)$ by taking the fibrewise direct sum. This behaves nicely with respect to the Steenrod bijections as:

$$BO(j) \times BO(k) \hookrightarrow BO(j+k)$$
$$(\xi, \zeta) \longmapsto \xi \oplus \zeta$$

So $\xi \oplus \zeta : X \longrightarrow BO(j) \times BO(k) \subset BO(j+k)$ classifies.

This suggests forming a directed set $BO(n) \subset BO(n+1)$ by $\xi \mapsto \xi \oplus \varepsilon$. Hence

"stabilisation"

$$\boxed{\begin{aligned} \varinjlim BO(N) &=: BO \\ \varinjlim BSO(N) &=: BSO \end{aligned}}$$

Def'n Vector bundles ξ and ζ are stably isomorphic if $\exists j, j'$ s.t. there is a v.b. iso

$$\xi \oplus \varepsilon^{j} \cong \zeta \oplus \varepsilon^{j'}$$

Theorem There is a bijection

$$\left\{ \begin{array}{l} \text{Iso classes of stable} \\ \text{v.b. over } X \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Htpy classes of maps} \\ X \longrightarrow BO \end{array} \right\}$$

③ Normal Bundles + Surgery

Def'n An *immersion* is a diff'ble map $f: N^n \to M^m$ s.t.

$$df|_x : T_x N \to T_{f(x)} M$$

is injective $\forall x \in N$.

Choosing a metric allows us to construct a v.b. over N that is fibrewise the orthogonal complement of TN.

i.e. $T_x N \oplus (T_x N)^\perp \cong T_{f(x)} M$

We call this bundle $\nu_f : N \to BO(m-n)$, the *normal bundle of the immersion*. Note

$$TN \oplus \nu_f \cong f^* TM$$

A *framing of an immersion* is a framing of ν_f.

Remark We want to talk about normal bundle independently of immersions (like we do with tangent bundles).

Def'n Let $g: M^m \hookrightarrow S^{m+k}$ be an embedding s.t. $\nu_g \oplus TM \cong \varepsilon^{m+k}$. Then define

$$\nu_M := \nu_g \quad \text{the } \textit{normal bundle of } M$$

This is clearly not a unique bundle, but we can stabilise:

The *stable normal bundle of M* is the unique map

$$\nu_M : M \to BO$$

such that any embedding defining the normal bundle $\nu_M : M \to BO(k)$ (in the sense above) is a representative.

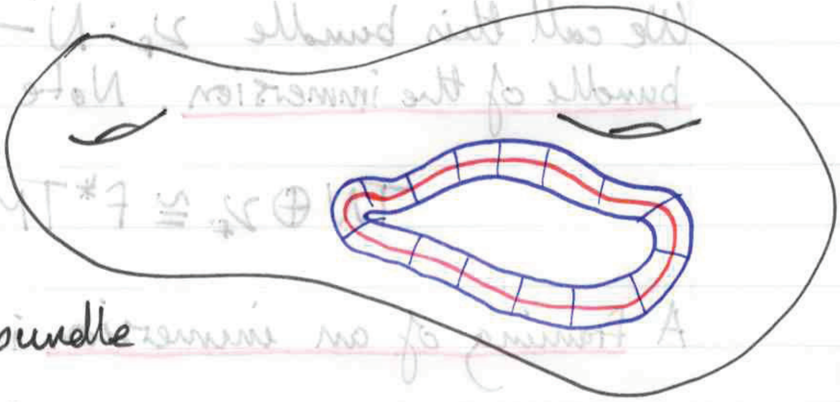
What has this all got to do with surgery?

Theorem (Tubular N'hood)

An embedding/immersion $f: N^n \to M^m$ extends to a codimension 0 embedding/immersion of the $(m-n)$-plane bundle ν_f:

$$E(\nu_f) \longrightarrow M.$$

In particular, if ν_f is trivial we can embed/immerse the disk bundle



$$D(\nu_f) \cong N \times D^{m-n} \longrightarrow M. \qquad \square$$

We get immediately:

Theorem

$x \in \pi_n(M)$ can be killed by an n-surgery $\iff$ x can be represented by an embedding with trivial normal bundle.

Corr If we are in the range of the Whitney Embedding Th'm i.e. $2n < m$.

$x \in \pi_n(M)$ can be killed $\iff (\nu_M)_* x = 0 \in \pi_n(BO)$.

<u>Proof</u> $2n < m \Rightarrow \exists$ an embedding $g : S^n \hookrightarrow M^m$

We know $\nu_g \oplus TS^n \cong g^* TM$

Add $\varepsilon \Rightarrow \nu_g \oplus (TS^n \oplus \varepsilon) \cong g^* TM \oplus \varepsilon$
$\cong g^*(TM \oplus \varepsilon)$

[<u>EXERCISE</u> $TS^n \oplus \varepsilon \cong \varepsilon^{n+1}$]

$$\Rightarrow \quad \nu_g \oplus \varepsilon^{n+1} \cong g^*(TM \oplus \varepsilon)$$

Add $g^* \nu_M \Rightarrow (\nu_g \oplus g^* \nu_M) \oplus \varepsilon^{n+1} \cong g^*(TM \oplus \varepsilon) \oplus g^* \nu_M$
$\cong g^*((TM \oplus \nu_M) \oplus \varepsilon)$
$\cong$ trivial

$\Rightarrow \nu_g \oplus g^* \nu_M$ is stably trivial

So vanishing $g^* \nu_M = (\nu_M)_* x \in \pi_n(BO)$ is equivalent to vanishing $\nu_g \in \pi_n(BO)$. But this is equivalent to $\nu_g = 0 \in \pi_n(BO(m-n))$ as $\pi_n(BO(m-n)) = \pi_n(BO)$ (i.e. we are in the stable range).

$\square$

④ Examples

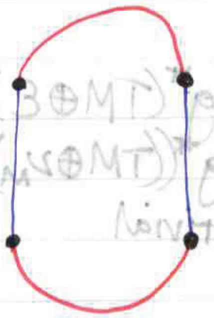
Embeddings in Spheres

$$S^m = \partial(D^{n+1} \times D^{m-n}) \qquad \forall n \leq m$$
$$= (S^n \times D^{m-n}) \cup (D^{n+1} \times S^{m-n-1})$$

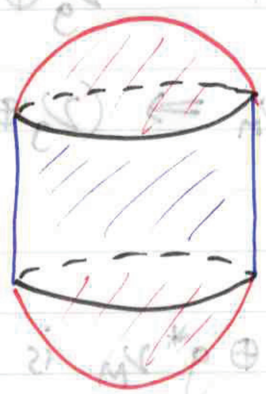
Glue along $S^n \times S^{m-n-1}$

EXAMPLE

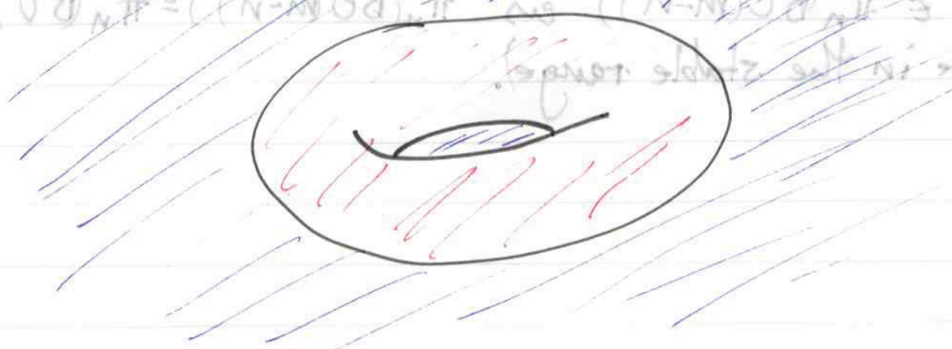
$m = 1,\ n = 0$



$m = 2,\ n = 0$



$m = 3,\ n = 1$



Choice of embedding

We have no choice in the embedding of the core S^n if we want it to represent some $x \in \pi_n(M^m)$.

However we have a choice of framing. This is equivalent to a choice of gluing along $S^n \times S^{m-n-1}$ where we fix the S^n factor. i.e. it is a map

$$\omega : S^n \longrightarrow \mathrm{Aut}(S^{m-n-1}) \cong O(m-n)$$

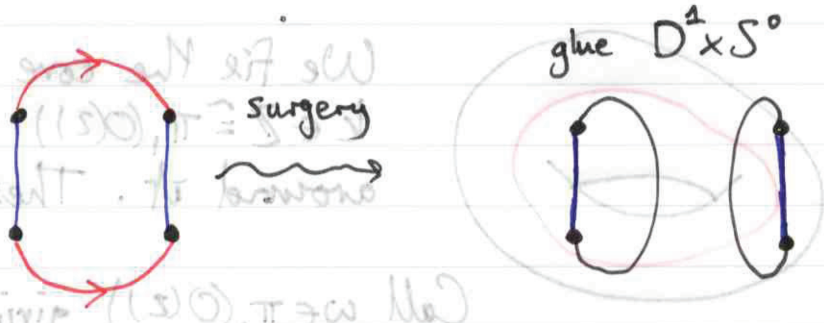
$\Rightarrow$ choose an element of $\pi_n(O(m-n))$ to get

$$g_\omega : S^n \times D^{m-n} \hookrightarrow (S^n \times D)^{m-n}) \cup_\phi (D^{n+1} \times S^{m-n-1})$$

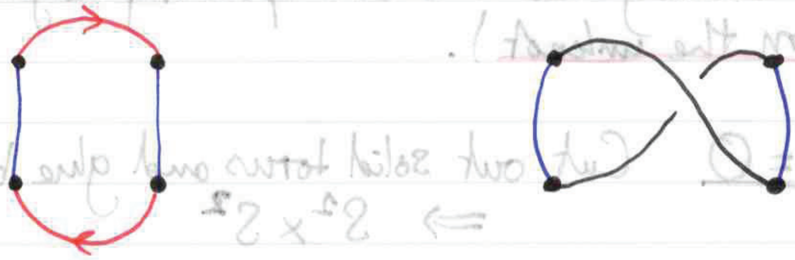
where $\phi(x,y) = (x, \omega(x)\, y)$ on the boundary.

<u>EXAMPLE</u> $m=1,\ n=0 \Rightarrow \pi_0(O(1)) = \mathbb{Z}_2$

ω trivial :



ω non-triv :



<u>Question</u> How does this relate to spin structures? The non-triv spin structure on S^1 extends to D^2, the triv one does not. How can we interpret it like this?

<u>EXERCISE</u> Recreate the Klein bottle / Torus example using this language.

<u>Proposition</u> (5.68 in Andrew Ranicki's book)

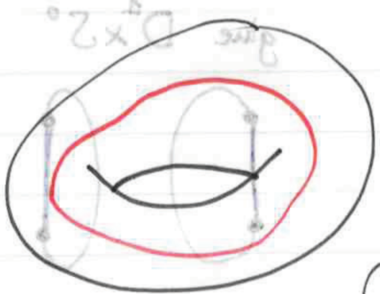
The effect of n-surgery on the embedding with framing ω given above is the sphere bundle over S^{n+1} glued by ω:

$$S(\omega) = (D^{n+1} \times S^{m-n-1}) \cup_\omega (D^{n+1} \times S^{m-n-1})$$

$\square$

This has a very nice consequence for S^2:

<u>EXAMPLE</u> $m=3$, $n=1$ $\Rightarrow$ $\pi_1(O(2)) = \mathbb{Z}$.



We fix the core torus and twist $k \in \mathbb{Z} \cong \pi_1(O(2))$ times as we go around it. These are Dehn twists!

Call $\omega \in \pi_1(O(2))$ giving $k \in \mathbb{Z}$ the k-framing and the corresponding surgery the <u>k-surgery (on the unknot)</u>.

<u>$k=0$</u> Cut out solid torus and glue backwards
$\Rightarrow S^1 \times S^2$

<u>$k=1$</u> Trivial framing $\Rightarrow$ effect is S^3
$\Rightarrow$ Hopf bundle!

Generally we will get the sphere bundle of $O(k) \to S^2$.

<u>EXERCISE</u>
- Calculate $\pi_1(O(k))$
- Show $O(1)$ is the universal cover of $O(k)$ $(k \neq 0)$.
- Describe the deck transformations.