Cobordism Theory

Carmen Rovi

Monday, 19 September 2011

Definition of cobordism

A **cobordism** of closed m-dimensional manifolds M_0^m and M_1^m is a $(m+1)$-dimensional manifold W^{m+1} with boundary

$$\partial W = \bar{M}_0 \sqcup M_1,$$

where $\bar{M}_0$ denotes M_0 with reverse orientation.

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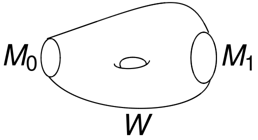

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Properties of the cobordism relation

Denoting the Cobordism relation as $\sim$, we define the following properties:

(i) $\sim$ is an equivalence relation,

(ii) $M \sim N$ implies $\partial M \cong \partial N$,

(iii) For all manifolds M, $\partial M \sim \emptyset$,

(iv) $M_1 \sim M_2$ and $N_1 \sim N_2 \Longrightarrow M_1 + N_1 \sim M_2 + N_2$.

The equivalence classes Ω_n of n-dimensional manifolds form an abelian group.

The direct sum of these abelian groups Ω_n of cobordism classes $[M]$ of n-dimensional manifolds,

$$\Omega_* = \bigoplus_{n \geq 0} \Omega_n,$$

form a graded, commutative ring.

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...But how do we compute cobordism groups?

Grassmann manifolds

$$Gr(r, n)$$

is the Grassmann manifold consisting of unoriented r-planes through the origin in $\mathbb{R}^{r+n}$.

$$Gr(r, n) \xrightarrow{i} Gr(r, n+1) \longrightarrow \ldots \longrightarrow Gr(r, \infty) = BO_r.$$

Canonical bundles of $Gr(r, n)$:

- The universal bundle $\gamma_{r,n}$
- Its orthogonal complement $\gamma_{r,n}^{\perp}$.

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Cobordism theory

Pullback constructions

$$\begin{array}{ccc} i^*\tau(\mathbb{R}^{n+r}) & \longrightarrow & \tau(\mathbb{R}^{n+r}) \\ \downarrow & & \downarrow \\ M & \xrightarrow{\ i\ } & \mathbb{R}^{n+r} \end{array}$$

similarly,

$$\begin{array}{ccc} \nu(M) & \longrightarrow & \gamma_{r,n} \\ \downarrow & & \downarrow \\ M & \longrightarrow & Gr(r,n) \end{array}$$

Classifying map

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Cobordism theory

(B_r, f_r) structure

Let $\nu(i) : M \longrightarrow BO_r$ be the classifying map of the normal bundle ν. A (B_r, f_r) structure on the normal bundle is defined when for the following diagram it holds that $\nu \simeq f_r \circ \widetilde{\nu}$, i.e. when the diagram commutes,

$$\begin{array}{ccc} & & B_r \\ & \overset{\tilde{\nu}}{\nearrow} & \big\downarrow{\scriptstyle f_r} \\ M & \xrightarrow[\nu(i)]{} & BO_r \end{array}$$

Moreover, any two liftings $\widetilde{\nu_1}$ and $\widetilde{\nu_2}$ are equivalent if they are homotopic. That is, if there exists a map $H : M \times I \longrightarrow B_r$ such that $H|_{M\times\{0\}} = \widetilde{\nu_1}$, $H|_{M\times\{1\}} = \widetilde{\nu_2}$ and $f_r \circ H(m, t) = \nu(m)$ for all $m \in M$ and $t \in I$.

(B, f) structures

$$
\begin{array}{ccccc}
 & & B_r & \xrightarrow{g_r} & B_{r+1} \\
 & \overset{\tilde{\nu}}{\nearrow} & \downarrow f_r & & \downarrow f_{r+1} \\
M & \xrightarrow[\nu(i)]{} & BO_r & \xrightarrow{j_r} & BO_{r+1}
\end{array}
\Longrightarrow
\begin{array}{ccc}
 & & B_{r+1} \\
 & \overset{\tilde{\nu}\circ g_r}{\nearrow} & \downarrow f_{r+1} \\
M & \xrightarrow[j_r\circ\nu(i)]{} & BO_{r+1}
\end{array}
=
\begin{array}{ccc}
 & & B_{r+1} \\
 & \overset{\tilde{\nu}(i')}{\nearrow} & \downarrow f_{r+1} \\
M & \xrightarrow[\nu(i')]{} & BO_{r+1}
\end{array}
$$

Such a sequence of (B_r, f_r) structures defines a (B, f) structure.

Definition (Thom space)

Let ξ be a vector bundle.
The total space of this vector bundle by $E(\xi)$.
Consider a subset of this total space A consisting of all the vectors of length at least one, i.e. $A = \{v \in E(\xi) : |v| \geq 1\}$.

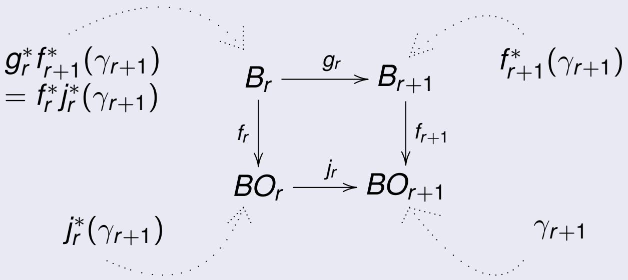
Then the **Thom space** of ξ, $T(\xi)$, is obtained by collapsing the whole of A to a point, which we will denote by ∞.

Aplying the Thom space construction to the first commutative diagram gives us the maps of Thom spaces as shown,

$$\begin{array}{ccc} B_r & \xrightarrow{g_r} & B_{r+1} \\ {\scriptstyle f_r}\downarrow & & \downarrow{\scriptstyle f_{r+1}} \\ BO_r & \xrightarrow{j_r} & BO_{r+1} \end{array} \quad \Longrightarrow \quad \begin{array}{ccc} \Sigma TB_r & \xrightarrow{Tg_r} & TB_{r+1} \\ {\scriptstyle \Sigma Tf_r}\downarrow & & \downarrow{\scriptstyle Tf_{r+1}} \\ \Sigma TBO_r & \xrightarrow{Tj_r} & TBO_{r+1} \end{array}$$

Cobordism theory

Vector bundles



The suspension map Σ induces the following map between homotopies,

$$\Sigma : \pi_{n+r}(TB_r, \infty) \longrightarrow \pi_{n+r+1}(\Sigma TB_r, \infty).$$

The map $Tg_r : \Sigma TB_r \longrightarrow TB_{r+1}$ induces:

$$Tg_r : \pi_{n+r+1}(\Sigma TB_r, \infty) \longrightarrow \pi_{n+r+1}(TB_{r+1}, \infty).$$

Composing these two maps we obtain,

$$Tg_r \circ \Sigma : \pi_{n+r}(TB_r, \infty) \longrightarrow \pi_{n+r+1}(TB_{r+1}, \infty),$$

and from this map we can define the stable homotopy group,

$$\lim_{r \to \infty} \pi_{n+r}(TB_r, \infty).$$

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Thom Cobordism Theorem

Let Ω_n be the cobordism group of n-dimensional (B, f) manifolds. Then,

$$\Omega_n(B, f) \cong \lim_{r\to\infty} \pi_{n+r}(TB_r, \infty).$$

...but first...

Time for a break!

Want to prove:

$$\Omega_n(B,f) \cong \lim_{r\to\infty} \pi_{n+r}(TB_r,\infty).$$

Vital elements in the proof:

- Definition of transversality
- Sard-Thom transversality theorem
- Pontryagin-Thom construction

Transversality

Let g be the map $g : N^n \to T(\xi)$, where $T(\xi)$ is the Thom space of r-bundle $\xi : X \to BO_r$.
Then g is **transverse** at the zero section $X \hookrightarrow T(\xi)$ if the inverse image is a closed $(n-r)$-dimensional submanifold,

$$M^{n-r} = g^{-1}(X) \subseteq N$$

Sard-Thom Transversality Theorem

Every continuous map $N^n \to T(\xi)$ from an n-dimensional manifold to the Thom space $T(\xi)$ of the r-bundle $\xi : X \to BO_r$ is homotopic to a map $g : N^n \to T(\xi)$, which is transverse at the zero section $X \hookrightarrow T(\xi)$.

First step: We first consider the map $\Theta : \Omega_n(B, f) \longrightarrow \lim_{n\to\infty} \pi_{n+r}(TB_r, \infty)$. Our first goal is to show that the map Θ is well defined and to describe an element of $\lim_{n\to\infty} \pi_{n+r}(TB_r, \infty)$.

- Consider a cobordism class $[M] \in \Omega_n$.
- Let $i : M \longrightarrow \mathbb{R}^{n+k}$, be an embedding.
- The classifying map of the normal bundle is then given by $\nu(i) : M \longrightarrow BO_r$.
- We denote the total space of this bundle by N and the projection map by $\pi : N \longrightarrow M$.

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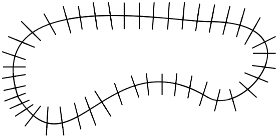
Choose a tubular neighbourhood N_ϵ of M.



We now write S^{n+r} as $\mathbb{R}^{n+r} \cup \infty$ and consider the map

$$c : S^{n+r} \longrightarrow N_\epsilon / \partial N_\epsilon,$$

We also define the map,

$$\epsilon^{-1} : N_\epsilon / \partial N_\epsilon \longrightarrow Th(\nu M).$$

Composing these maps,

$$\mathbb{R}^{n+r} \cup \infty = S^{n+r} \xrightarrow{c} N_\epsilon / \partial N_\epsilon \xrightarrow{\epsilon^{-1}} Th(\nu M).$$

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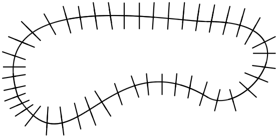

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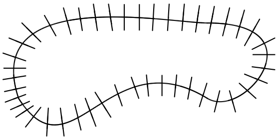

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So we have,

$$\mathbb{R}^{n+r} \cup \infty = S^{n+r} \xrightarrow{c} N_\epsilon/\partial N_\epsilon \xrightarrow{\epsilon^{-1}} Th(\nu M).$$

Composing this with the natural inclusions of Grassmann manifolds, we obtain,

$$S^{n+r} \xrightarrow{c} N_\epsilon/\partial N_\epsilon \xrightarrow{\epsilon^{-1}} Th(\nu M) \longrightarrow Th(\gamma_{r,n}) \longrightarrow \ldots \longrightarrow Th(\gamma_r) = TBO_r.$$

Definition of the element θ

So we have now defined a map $\theta : S^{n+r} \longrightarrow TBO_r$, and hence we have an element of $\lim_{n\to\infty} \pi_{n+r} TBO_r$.

Second step: We now want to show that the map $\Theta : \Omega_n(B, f) \to \lim_{n\to\infty} \pi_{n+r}(TB_r, \infty)$ is an homomorphism. So we want to show that $\Theta([M_1] + [M_2]) = \Theta([M_1]) + \Theta([M_2])$.

- Choose $[M_1], [M_2] \in \Omega_n(B, f)$ and embeddings $i_1 : M_1 \longrightarrow \mathbb{R}^{n+r}$ and $i_2 : M_2 \longrightarrow \mathbb{R}^{n+r}$ which send M_1 and M_2 into different half planes.
- Also choose tubular neighbourhoods N_ϵ^1 of M_1 and N_ϵ^2 of M_2

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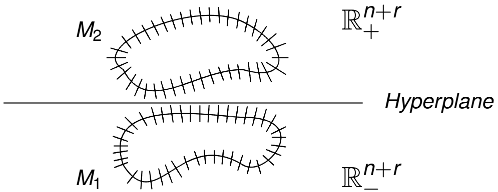

We obtain the map,

$$S^{n+r} \xrightarrow{\text{collapsing}} S^{n+r} \vee S^{n+r} \xrightarrow{\Theta[M_1]\vee\Theta[M_2]} TB_r.$$

This is the same map as defined by the sum of homotopy classes:

$$\Theta[M_1] + \Theta[M_2] \Longrightarrow \Theta([M_1] + [M_2]) = \Theta[M_1] + \Theta[M_2]$$

Hence Θ is a homomorphism.

Surjective

Third step: We now Show that

$$\Theta : \Omega_n(B, f) \longrightarrow \lim_{n \to \infty} \pi_{n+r}(TB_r, \infty)$$

is surjective,

We start by noting that S^{n+r} is compact, so for any open cover we can find a finite subcover. That is, for some s, we have $Tf_r \circ \theta(S^{n+r}) \subset Th(\gamma_{r+s})$.

So we homotope the map $Tf_r \circ \theta : S^{n+r} \longrightarrow Th(\gamma_{r,n})$ to a new (homotopic) map $\tilde{\theta}$ such that:

(i) $\tilde{\theta}$ is differentiable on a neighbourhood of $Gr(r,s)$.

(ii) $\tilde{\theta}$ is transverse regular on $Gr(r,s)$, which is the zero section of $\gamma_{r,s}$. This implies that $\tilde{\theta}^{-1}(Gr(r,s))$ is a manifold, i.e. $M = \tilde{\theta}^{-1}(Gr(r,s))$. The dimension of M is $\dim M = \dim \tilde{\theta}^{-1}(Gr(r,s)) = n$, since M has the same codimension as $Gr(r,s)$ in $\mathbb{R}^{n+r}$.

(iii) Evaluating the map $\tilde{\theta}^{-1}$ on a small neighbourhood of $Gr(r,s)$ gives us a tubular neighbourhood of M.

Injective

Fourth step: To finish the proof we will show that the map Θ is injective.

Let M be a (B, f) manifold, such that $M \in \mathrm{Ker}(\Theta)$. Then for some r, the map $\Theta(M) : S^{n+r} \longrightarrow TB_r$ is homotopic to the trivial map $t : S^{n+r} \longrightarrow \infty$.

By compactness of S^{n+r}, we have $S^{n+r} \xrightarrow{L} TB_r \xrightarrow{Tf_r} TBO_r$, so that,

$$S^{n+r} \times [0,1] \subset Th(\gamma_{r,s}) \quad \text{for some } s \geq n.$$

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We now want to "deform" or homotope $Tf_r \circ L$ to a map H_r in a neighbourhood of $Gr(r,s)$. This map H_r satisfies the following properties:

(i) H_r is differentiable on a neighbourhood of $Gr(r,s)$.

(ii) H_r is transverse on $Gr(r,s)$.

This implies that,

(1) $W = H_r^{-1}(Gr(r,s))$ is a manifold, in fact a submanifold of $\mathbb{R}^{n+r} \times [0,1]$.

(2) $\partial W = W \cap \partial(S^{n+r} \times [0,1]) = M$ and $\partial W \subseteq \mathbb{R}^{n+r} \times 0$.

(3) $H_r|_W$ is the normal map $W \xrightarrow{\nu = H_r|_W} TBO_r$.

By the covering homotopy theorem we deduce that $[M]$ is the zero class of $\Omega_n(B,f)$.

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So we have now proved the Thom Cobordism Theorem,

$$\Omega_n(B, f) \cong \lim_{r \to \infty} \pi_{n+r}(TB_r, \infty).$$

Thank you for listening!