Cobordism Theory and the Signature Theorem

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Characteristic classes are cohomology classes. Here we are going to consider four important types of characteristic classes, namely:

Characteristic class	**Notation**	**Vector bundle** $E \longrightarrow B$
Stiefel-Whitney classes	$w_i(\xi) \in H^i(B(\xi); \mathbb{Z}_2)$	real vector bundle
Chern classes	$c_i(\xi) \in H^{2i}(B(\xi); \mathbb{Z})$	complex vector bundle
Pontrjagin classes	$p_i(\xi) \in H^{4i}(B(\xi); \mathbb{Z})$	real vector bundle
Euler class	$e(\xi) \in H^n(B(\xi); \mathbb{Z})$	Oriented, $n - dim$, real vector bundle

Characteristic classes

What is a characteristic class?

Every cohomology class $c \in H^*(BG; R)$ determines a characteristic class of principle G-bundles over the base space M.
Infinite Grassmann manifolds $Gr(n, \infty)$ are classifying spaces for linear groups.

If $f : \xi \longrightarrow \gamma_n$ is any bundle map, then this map induces a unique homotopy class of maps of base spaces.

We can also write $f_\xi^*(\gamma_n) = \xi$. So if $c(\gamma_n) \in H^*(BO_n; R)$, then this cohomology class can be pulled back by the map f_ξ^* so that,

$$f_\xi^* c(\gamma_n) = c(f_\xi^*(\gamma_n)) = c(\xi) \in H^*(M; R).$$

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$$\boxed{f_\xi^* c(\gamma_n) = c(f_\xi^*(\gamma_n)) = c(\xi) \in H^*(M; R).}$$

(a) **Existence axiom:** For each real vector bundle ξ there exists a sequence of Stiefel-Whitney cohomology classes

$$w_i(\xi) \in H^i(B(\xi); \mathbb{Z}_2), \text{where } i = 0, 1, \ldots$$

For $i = 0$, the Stiefel-Whitney class is the unit element in $H^0(B(\xi); \mathbb{Z}_2)$.

(b) **Naturality axiom:** Let $f^*(\xi)$ be a pullback bundle of the bundle ξ, then,

$$w_i(f^*(\xi)) = f^*(w_i(\xi)).$$

(c) **The Whitney product axiom:** Let ξ and ξ' be two different vector bundles over the same base space B, then,

$$w_1(\xi \oplus \xi') = w_1(\xi) + w_1(\xi'),$$

$$w_2(\xi \oplus \xi') = w_2(\xi) + w_1(\xi)w_1(\xi') + w_2(\xi'),$$

and, in general, $$w_k(\xi \oplus \xi') = \sum_{i=0}^{k} w_i(\xi) \cup w_{k-i}(\xi').$$

(d) $w_i(\xi)$ **as a generator of** $H^i(\mathbb{R}P^\infty; \mathbb{Z}_2)$**:** For the line bundle $E \to \mathbb{R}P^\infty$, $w_i(\xi)$ is a generator of $H^i(\mathbb{R}P^\infty; \mathbb{Z}_2)$.

The cohomology ring $H^*(BO_n; \mathbb{Z}/2)$ is,

$$H^*(BO_n; \mathbb{Z}_2) = \mathbb{Z}_2[w_1(\gamma^n), \ldots, w_n(\gamma^n)].$$

Note that there are no polynomial relations between the generating Stiefel-Whitney classes $w_1(\gamma^n), \ldots, w_n(\gamma^n)$.

The cohomology ring $H^*(BSO_n; \mathbb{Z}/2)$ of is,

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Stiefel-Whitney numbers

Definition

Stiefel-Whitney numbers Let $I = (i_1, \ldots, i_r)$ be any partition of n, and τ be the tangent bundle of M^n. Then the Stiefel-Whitney number of M associated with the monomial $w_{i_1}(\tau) \ldots w_{1_r}(\tau)$ is given by,

$$\left\langle w_{i_1}(\tau) \ldots w_{i_r}(\tau), \mu_M \right\rangle = w_{i_1} \ldots w_{i_r}[M^n] \in \mathbb{Z}_2.$$

Stiefel-Whitney numbers and cobordism

- If M^n is the boundary of an $(n+1)$-dimensional manifold, then the Stiefel-Whitney numbers of M are all zero.
- Conversely, if all the Stiefel-Whitney numbers of a manifold are zero, then M is a boundary.

Let M and N be two closed, smooth, n-dimensional manifolds, then M and N belong to the same cobordism class if and only if their Stiefel-Whitney numbers are equal.

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Pontryagin classes

Pontryagin classes are cohomology classes: $p_i(\xi) \in H^{4i}(B;\mathbb{Z})$.

The i-th Pontryagin class $p_i(\xi) \in H^{4i}(B;\mathbb{Z})$ is defined in terms of Chern classes as

$$p_i(\xi) = (-1)^i c_{2i}(\xi \otimes \mathbb{C}).$$

The **total** Pontryagin class is defined as,

$$p(\xi) = 1 + p_1(\xi) + \cdots + p_{[n/2]}(\xi).$$

This class is a unit in the cohomology ring $H^*(B;\mathbb{Z})$.

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Pontryagin classes satisfy certain properties:

(a) The naturality property is satisfied. That is, if $f^*(\xi)$ be a pullback bundle of the bundle ξ, then,

$$p_i(f^*(\xi)) = f^*(p_i(\xi)).$$

(b) If ϵ^n is a trivial n-plane bundle, then $p(\xi \oplus \epsilon^n) = p(\xi)$.

(c) A product formula for $p(\xi \oplus \nu)$ in the case of Pontryagin classes is,

$$p(\xi \oplus \nu) \equiv p(\xi)p(\nu) \mod \text{(elements of order 2)},$$

or equivalently, $2p(\xi \oplus \nu) = 2p(\xi)p(\nu)$.

(d) Let ξ be a $2n$-plane vector bundle, then

$$p_n(\xi) = e^2(\xi).$$

Pontryagin numbers

Pontryagin numbers

Let M^{4n} be a smooth, compact, oriented manifold with tangent bundle denoted by τ^{4n}, and let $I = i_1, \ldots, i_r$ be a partition of n, then the I-th Pontryagin number is an integer defined as

$$p_I[M^{4n}] = p_{i_1} \ldots p_{i_r}[M^{4n}] = \left\langle p_{i_1}(\tau^{4n}) \ldots p_{i_r}(\tau^{4n}), \mu_{4n} \right\rangle .$$

where $\mu_{4n} \in H^{4n}(M^{4n}; \mathbb{Z})$ is the fundamental or orientation class of the manifold M^{4n}.

Pontryagin numbers projective space

The Pontryagin numbers of complex projective spaces $\mathbb{C}P^{2n}$. The complex manifold $\mathbb{C}P^{2n}$ has real dimension $4n$, so if $i_1, \ldots, i_r$ is a partition of n, then

$$p_{i_1} \ldots p_{i_r}[\mathbb{C}P^{2n}] = \begin{pmatrix} 2n+1 \\ i_1 \end{pmatrix} \ldots \begin{pmatrix} 2n+1 \\ i_r \end{pmatrix}.$$

A relevant issue in the context of oriented cobordism is to consider how Pontryagin classes and Pontryagin numbers may change when the orientation of the manifold is reversed.

Pontryagin classes and diffeomorphisms

If we reverse the orientation of a manifold M^{4n}:

- Pontryagin classes remain unchanged,
- Pontryagin numbers change sign.

As a consequence if some Pontryagin number is non-zero then M^{4n} does not posses any orientation reversing diffeomorphism.

The Pontryagin numbers of $\mathbb{C}P^{2n}$ are non-zero. Consequently, $\mathbb{C}P^{2n}$ does **not** have any orientation reversing diffeomorphism.

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If some Pontryagin number of a manifold M^{4n} is non-zero, then M^{4n} is not the boundary of a smooth, compact oriented $(4n+1)$-dimensional manifold.

The Pontryagin numbers of $\mathbb{C}P^{2n}$ are non-zero, so that $\mathbb{C}P^{2n}$ cannot be an oriented boundary

Consequences :

The oriented cobordism group Ω_n is finite for $n \neq 4k$, and is a finitely generated group with rank equal to $p(k)$, the number of partitions of k, when n = 4k.

The tensor product $\Omega_* \otimes \mathbb{Q}$ is a polynomial algebra over $\mathbb{Q}$ with independent generators $\mathbb{C}P^2, \mathbb{C}P^4, \ldots, .$

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Computation of Cobordism groups

n	$\mathfrak{N}_n$	$\Omega_n \otimes \mathbb{Q}$	Ω_n
0	$\mathbb{Z}_2$	$\mathbb{Q}$	$\mathbb{Z}$
1	0	0	0
2	$\mathbb{Z}_2$	0	0
3	0	0	0
4	$(\mathbb{Z}_2)^2$	$\mathbb{Q}$	$\mathbb{Z}$
5	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$
6	$(\mathbb{Z}_2)^3$	0	0
7	$\mathbb{Z}_2$	0	0
8	$(\mathbb{Z}_2)^5$	$(\mathbb{Q})^2$	$(\mathbb{Z})^2$
9	$(\mathbb{Z}_2)^3$	0	$(\mathbb{Z}_2)^2$
10	$(\mathbb{Z}_2)^8$	0	$\mathbb{Z}_2$
11	$(\mathbb{Z}_2)^4$	0	$\mathbb{Z}_2$
12	$(\mathbb{Z}_2)^7$	$(\mathbb{Q})^3$	$(\mathbb{Z}_2)^3$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

Break!

Let $\mathcal{A}$ be a commutative ring consisting of all formal sums $a_0 + a_1 + a_2 + \dots$.

Consider a sequence of polynomials denoted by

$$K_1(x_1), K_2(x_1, x_2), K_3(x_1, x_2, x_3), \dots,$$

Each of the x_i is assigned degree $i \implies K_n(x_1, x_2, \dots, x_n)$ is a polynomial homogeneous of degree.

Hirzebruch Signature Theorem

For each element in the ring $\mathcal{A}$ with leading term 1, i.e., $a = 1 + a_1 + a_2 + \cdots \in \mathcal{A}$, we define $K(a) \in \mathcal{A}$ by,

$$K(a) = 1 + K_1(a_1) + K_2(a_1, a_2) + K_3(a_1, a_2, a_3) + \dots$$

Definition

The polynomials K_n form a **multiplicative sequence** if for $a, b \in \mathcal{A}$ with leading term 1, the relation

$$K(ab) = K(a)K(b),$$

is satisfied.

Hirzebruch Signature Theorem

Multiplicative sequences can be used to define the relations between characteristic classes.

These sequences are in one-one relation to certain formal power series.

power series and multiplicative sequence

Let $f(t) = 1 + \lambda_1 t + \lambda_2 t^2 + \ldots$ be a formal power series with coefficients in $\mathbb{Q}$, then there exists one multiplicative sequence K_n such that the coefficient of x_1^n in each polynomial $K_n(x_1, \ldots, x_n)$ is λ_n.

Note that this is the same as saying that $K(1 + t) = f(t)$.

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Genus

Let $K_n(x_1, \dots, x_n)$ be a multiplicative sequence, then we define the **K − genus**, $K[M^n]$, as follows,

(i) If $n \neq 4k$ then $K[M^n] = 0$,

(ii) If $n = 4k$, then $K_n[M^{4k}] = \langle K_n(p_1, \dots, p_n), [M^{4k}] \rangle \in \mathbb{Q}$.

where p_i denotes the ith-Pontryagin class.

Towards the definition of the $L-genus$

- Consider the formal power series

$$\begin{aligned} f(t) &= \sqrt{t}/\tanh\sqrt{t} \\ &= 1 + \frac{1}{3}t - \frac{1}{45}t^2 + \frac{1}{945}t^3 - \cdots + (-1)^{k-1}2^{2k}B_k t^k/(2k)! + \ldots, \end{aligned}$$

  where B_k is the k-th Bernoulli number.
- Also consider the multiplicative sequence such that the coefficient of p_1^n in each polynomial $L_n(p_1, \ldots, p_n)$ is given by the coefficient of t^n in $f(t) = \sqrt{t}/\tanh\sqrt{t}$.

The $L-$ *genus*

The **L − genus**, $L[M^n]$ is defined as

(i) If $n \neq 4k$ then $L[M^n] = 0$,
(ii) If $n = 4k$, then $L_n[M^{4k}] = \langle L_n(p_1, \dots, p_n), [M^{4k}] \rangle \in \mathbb{Q}$.

Hirzebruch Signature Theorem

Following the above definition, we can write several L-polynomials as,

$$\begin{array}{rcl} L_1(p_1) & = & \frac{1}{3}p_1, \\ L_2(p_1,p_2) & = & \frac{1}{45}(7p_2 - p_1^2), \\ L_3(p_1,p_2,p_3) & = & \frac{1}{945}(62p_3 - 13p_1p_2 + 2p_3), \\ \vdots & = & \vdots \end{array}$$

Hirzebruch Signature Theorem

Definition

The **Todd-genus** is the genus which has multiplicative seguences corresponding to the power series $f(t) = \frac{t}{1-e^{-t}}$.

Definition

The $\hat{a}$ **genus** is the genus which has multiplicative seguences corresponding to the power series $f(t) = \frac{\frac{1}{2}\sqrt{x}}{\sinh\sqrt{x}}$.

Any genus K gives rise to a ring homomorphism

$$\begin{array}{ccc} \Omega_* & \longrightarrow & \mathbb{Q} \\ M & \longmapsto & K[M] \in \mathbb{Q}. \end{array}$$

Moreover this also provides an algebra homomorphism from $\Omega_* \otimes \mathbb{Q}$ to $\mathbb{Q}$.

If M and N are smooth, compact oriented manifolds, then

$$K[M \times N] = K[M]K[N] \quad \text{and} \quad K[M + N] = K[M] + K[N].$$

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The Signature

Consider a compact, oriented manifold M of dimension n. The signature $\sigma(M)$ is defined as follows,

(i) If $n \neq 4k$, then $\sigma(M) = 0$,

(ii) If $n = 4k$, then $\sigma(M)$ is the signature of the intersection form

$$\sigma(M) = \sigma(H^{2k}(M; \mathbb{R}), \lambda)$$

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...but what is an intersection form?

A symmetric form (K, λ) over $\mathbb{R}$ is a finite-dimensional real vector space K together with a bilinear pairing

$$\begin{array}{rccc} \lambda : & K \times K & \longrightarrow & \mathbb{R} \\ & (x, y) & \longmapsto & \lambda(x, y) \end{array}$$

The **intersection form** of a closed oriented $4k$-dimensional manifold M^{4k} is the nonsingular symmetric form over $\mathbb{R}$, $(H^{2k}(M; \mathbb{R}), \lambda)$ with

$$\begin{array}{rcccccc} \lambda : & H^{2k}(M; \mathbb{R}) & \times & H^{2k}(M; \mathbb{R}) & \longrightarrow & \mathbb{R} \\ & (\ x & , & y\ ) & \longmapsto & \lambda(x, y) = \langle x \cup y, [M] \rangle \end{array}$$

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- Choose an appropriate basis $a_1, \ldots, a_r$ for $H^{2r}(M^{4k};\mathbb{Q})$ so that the matrix with entries $\langle a_i \cup a_j, [M] \rangle$ is diagonal,
- then $\sigma(M)$ can be found by subtracting the number of negative entries from the number of positive entries.

Properties of the Signature

The signature satisfies the following properties,

(i) $\sigma(M + N) = \sigma(M) + \sigma(N)$,

(ii) $\sigma(M \times N) = \sigma(M)\sigma(N)$,

(iii) If M is an (oriented) boundary, this implies that $\sigma(M) = 0$.

Let M and N be two closed, oriented, cobordant $4k$-dimensional manifolds and W be an oriented cobordism between them. Then the signature is an oriented cobordism invariant, where,

$$\sigma(M) = \sigma(N).$$

The signature defines homomorphisms on the $4k$-dimensional oriented cobordism groups,

$$\begin{array}{rccc} \sigma: & \Omega_{4k} & \longrightarrow & \mathbb{Z} \\ & [M] & \longmapsto & \sigma(M). \end{array}$$

The signature also induces an algebra homomorphism $\Omega_* \otimes \mathbb{Q}$ to $\mathbb{Q}$.

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Recall the power series,

$$\begin{aligned} f(t) &= \sqrt{t}/\tanh\sqrt{t} \\ &= 1 + \tfrac{1}{3}t - \tfrac{1}{45}t^2 + \tfrac{1}{945}t^3 - \cdots + (-1)^{k-1}2^{2k}B_k t^k/(2k)! + \ldots \end{aligned}$$

where B_k is the k-th Bernoulli number.

Let $L_k(p_1, \ldots, p_k)$ be the multiplicative sequence of polynomials belonging to this power series.

The Signature Theorem

The signature is equal to the L-genus $L[M^{4k}] = \langle L_k(p_1, \ldots, p_k), [M^{4k}] \rangle$,

$$\sigma(M^{4k}) = L[M^{4k}].$$

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Key idea for the proof:

- $M \mapsto L(M)$ and $M \mapsto \sigma(M)$ give rise to algebra homomorphisms $\Omega_* \otimes \mathbb{Q} \to \mathbb{Q}$,
- We only need to check that it holds on generators of $\Omega_* \otimes \mathbb{Q}$.

$$\sigma(\mathbb{C}P^{2k}) = L[\mathbb{C}P^{2k}]. \tag{1}$$

Exercices

Determine the signature of $\mathbb{C}P^n \times \mathbb{C}P^n$ for any $n \in \mathbb{N}$

Show that $\sigma(M\#N) = \sigma(M)\#\sigma(N)$

Determine whether the 8 manifolds $\mathbb{C}P^2 \times \mathbb{C}P^2$ and $\mathbb{C}P^4$ are cobordant.

Show that $\mathbb{C}P^n \times \mathbb{C}P^m$ and $\mathbb{C}P^{n+m}$ are linearly independent elements of the corresponding oriented cobordism group, when n and m are even.

Thank you for listening!