The h-cobordism Theorem

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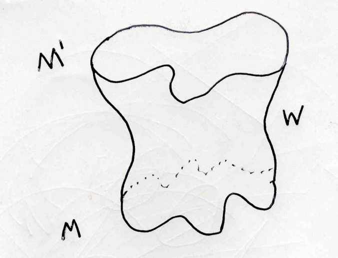
Introduction

Recall that an oriented *cobordism* between two smooth oriented m-dimensional manifolds M^m, M'^m is a smooth oriented $(m+1)$-dimensional manifold W^{m+1} such that the boundary ∂W splits as the disjoint union $\partial W = M \sqcup (-M')$.

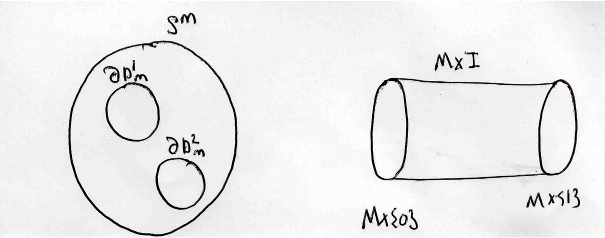


h-Codordism

Definition: A *h-cobordism* is a cobordism $(W; M, M')$ such that M and M' are deformation retracts of W. We say that M, M' are *h-cobordant* through W.

(i) Removing two disjoint discs $D_1^m, D_2^m \subset S^m$ we obtain a h-cobordism $(S^m; S^{m-1}, S^{m-1})$.

(ii) For a smooth manifold M we have the trivial cobordism $(M \times [0,1], M \times \{0\}, M \times \{1\})$



The h-Cobordism Theorem

Theorem: Let M^m, M'^m be compact, simply-connected, oriented m-dimensional manifolds which are h-cobordant through a simply-connected $(m+1)$-dimensional manifold W^{m+1}. For $m \geq 5$ there exists a diffeomorphism $W \cong M \times [0,1]$ which restricts to the identity from $M \subset W$ to $M \times \{0\} \subset M \times [0,1]$.

Corollary: Under the same conditions, M and M' are diffeomorphic.

High Dimensional Poincaré Conjecture

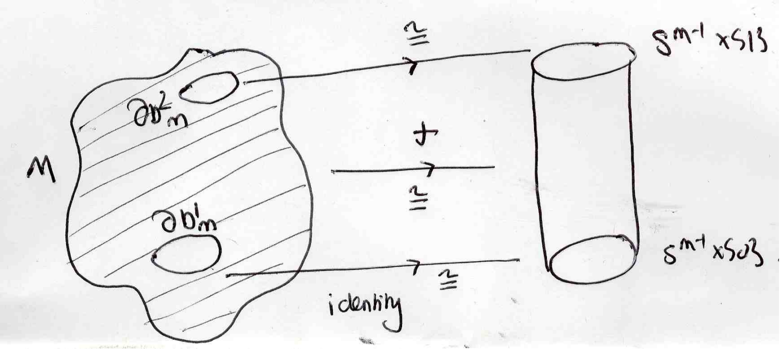
Claim: Let $m \geq 5$ and let M^m be a smooth m-dimensional manifold which is homotopy equivalent to S^m. Then M^m is homeomorphic to S^m.

Proof: We will first prove the claim for $m \geq 6$. We cut two small disks $D_1^m, D_2^m \subset M^m$ to obtain a h-cobordism $(M - (D_1 \cup D_2); S^{m-1}, S^{m-1})$. By the above theorem, there is a diffeomorphism

$$f : M - (D_1^m \cup D_2^m) \to S^{m-1} \times [0, 1]$$

where f acts as the identity on the lower boundary component $\partial D_1^m = S^{m-1}$.

Case $m = 6$



We wish to extend f over these two missing discs to obtain a homeomorphism $F : M \to S^m$. We may extend f over D_1^m by patching f with the identity map on D_1^m.

The restriction of f to $\partial D_2^m = S^{m-1} \to S^{m-1} \times \{1\}$ is a diffeomorphism. We may radially extend this to a homeomorphism of D_2^m with itself via the formula

$$F(rx) := rf(x), x \in \partial D_2^m, r \in [0, 1].$$

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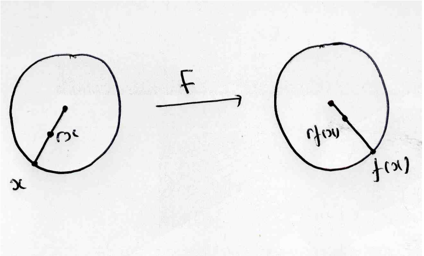


Patching these extension to f gives a homeomorphism between M and S^m.

Case $m = 5$

If we follow the same approach for the case $m = 5$ we cannot apply the h-cobordism theorem because we have a h-cobordism between two 4-dimensional manifolds.

Instead we use the fact that M exists as the boundary of a 6-dimensional manifold W. In fact, the homotopy equivalence $\partial W = M \simeq S^5 = \partial D^6$ can be extended to a homotopy equivalence $W \cong D^6$, so W is contractible.

We may cut out a disc D^6 from the interior of W to obtain a h-cobordism between S^5 and M. Then by the h-cobordism theorem M is diffeomorphic to S^5.

Handle Decompositions

Recall that every cobordism $(W; M, M')$ admits a Morse function $f : (W : M, M') \to [0,1]$ and the space of Morse functions on W is an open dense subset of the space of smooth functions on W. For a point $c \in [0,1]$ the ascending cobordism of f is $W_c := f^{-1}[0,c]$.

- If there are no critical values of f in $[c,c'] \subset [0,1]$ then W'_c is diffeomorphic to W_c.
- If there is a single critical point in $f^{-1}[c,c']$ of index k then $W_{c'}$ is diffeomorphic to W_c with a k-handle $h^k = D^k \times D^{m+1-k}$ attached.

We may assume that there are no critical points of f in $\partial W = M \cup M'$, otherwise we may perturb f slightly to a form a new Morse function with no critical points on the boundary. In particular there is a collar neighbourhood $M \times [0,\epsilon]$ of $M = f^{-1}(0) \subset W$ to which we can attach a series of handles in order to obtain W.

The Game Plan

The central idea is to find a Morse function with no critical point, or equivalently to find a handle decomposition of W with no handles.

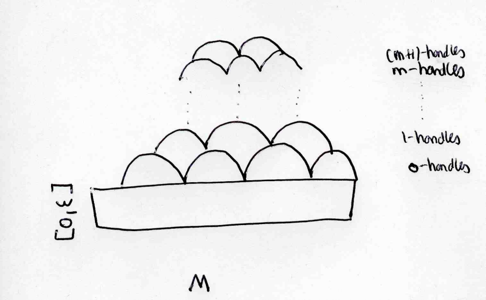
In this case we have a sequence of diffeomorphisms

$$W = W_1 \cong W_\epsilon \cong M \times [0, \epsilon] \cong M \times [0, 1]$$

whose composition restricts to the identity on $M \subset W$. We will show that we can slide, reorder, create and cancel handles until there are none left.

Simplification: Rearranging Handles

Claim: We may rearrange the handles so that they are attached in order of increasing dimension.

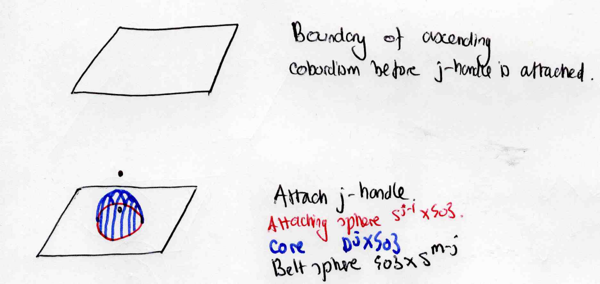


Theorem: Let $f : W \to \mathbb{R}$ be a Morse function. Then we can change f to a Morse function $g : W \to \mathbb{R}$ which is self indexing and which has the same set of critical points of f and the same Morse index as f for each critical point $p \in \text{Crit}(f) = \text{Crit}(g)$.

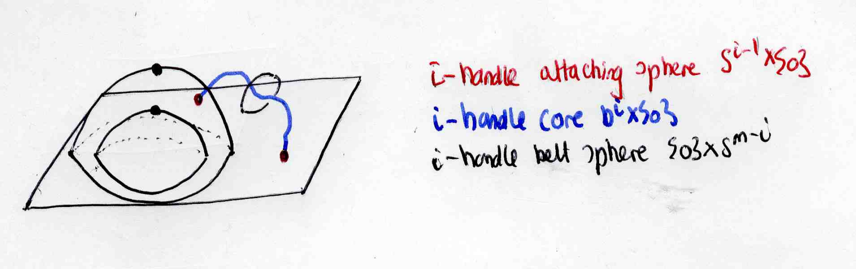
The Alternative

Claim: Whenever $i \leq j$, we may slide any i-handle off any j-handle it meets.

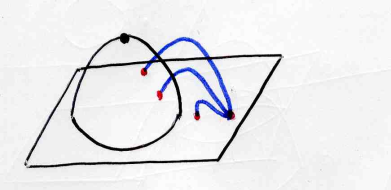
Idea: Suppose we have just attached a j-handle to the ascending cobordism W_c.



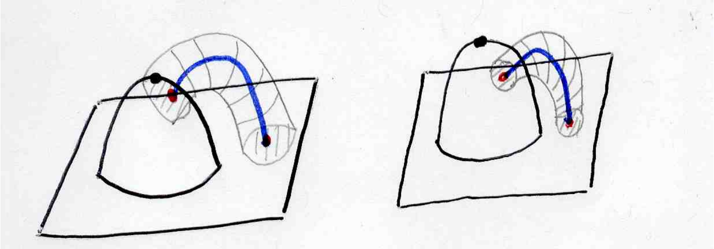
The i-handle is then attached to the new boundary component $D^j \times S^{m-j}$



Providing that the attaching sphere of the *i*-handle does not meet the belt sphere of the *j*-handle, we may radially move the *i*-handle off the *j*-handle.



We may be in the case where the i-attaching sphere does not meet the j-belt sphere, but the thickened attaching i-sphere $S^{i-1} \times D^{m-i+1}$ meets the j-belt sphere.



In this case we shrink the i-handle a little until it does not meet the j-belt sphere and then push radially.

Note that the i-attaching sphere and the j-belt sphere lie in an m-dimensional manifold which exists as the upper boundary of a particular $(m+1)$-dimensional ascending cobordism.

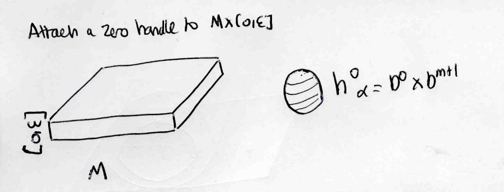
Generically, the intersection $S^{i-1} \cap S^{m-j}$ is of dimension

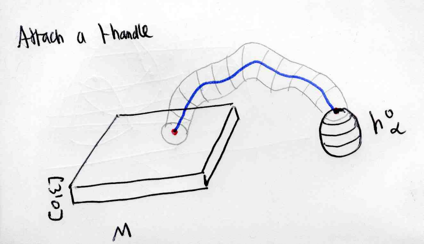
$$(i-1)+(m-j)-m=i-j-1.$$

We assumed that $i \leq j$, so generically S^{i-1} and S^{m-j} do not intersect.

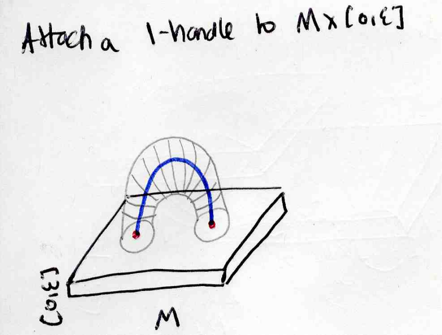
Handle Creation and Cancellation

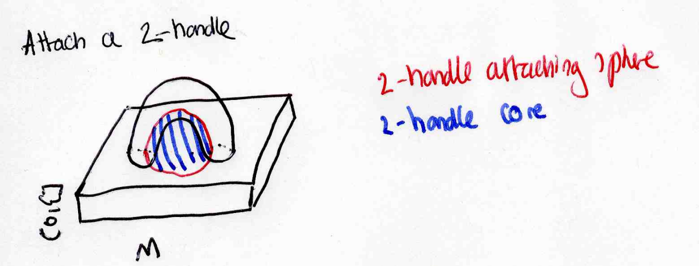
In certain cases when we attach a k handle we may leave a space which is later filled by a $(k+1)$-handle so that the two handles effectively cancel each other out.



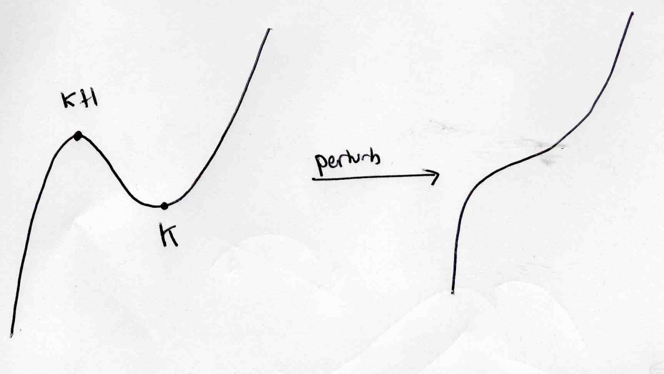


We can try a similar trick with a 1-handle and a 2-handle





This can also be done on the level of Morse functions



When can we guarantee that two handles cancel?

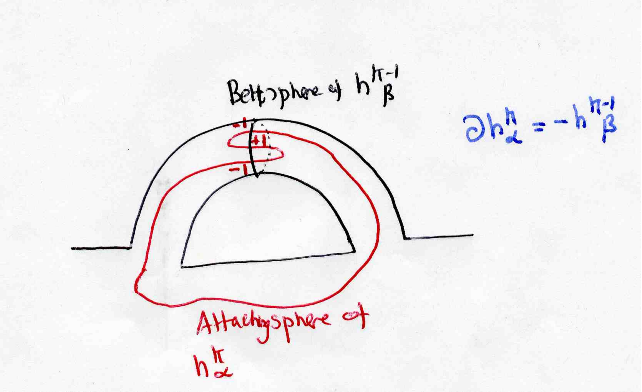
Handle Chain Complex

For a handle decomposition we have the handle chain complex (C, ∂).

The k-th chain group C_k is the free abelian group on the set of k-handles and the differential $\partial_k : C_k \to C_{k-1}$ is given by

$$\partial_k(h_\alpha^k) := \sum_{h_\beta^{k-1}} h_\beta^{k-1} < h_\alpha^k | h_\beta^{k-1} > h_\beta^{k-1}$$

The *incidence number* $< h_\alpha^k | h_\beta^{k-1} >$ is defined to be the intersection number of the attaching sphere of h_α^k and the belt sphere of h_β^{k-1}.

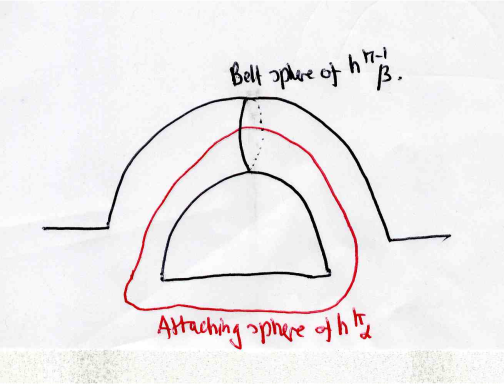


A necessary condition for cancellation of h_β^{k-1} with h_α^k is that

$$\partial_k h_\alpha^k = \pm h_\beta^{k-1}$$

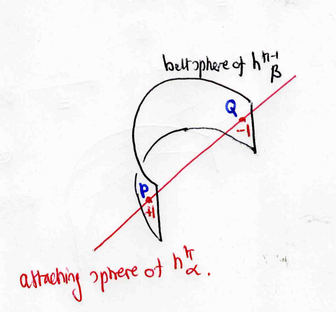
That is the attaching sphere of h_α^k has *algebraic* intersection number ± 1 with the belt sphere of h_β^{k-1} and has algebraic

For a sufficient condition we need more, namely the attaching sphere of h_α^k must *geometrically* intersect the belt sphere of h_β^{k-1} in exactly one point.



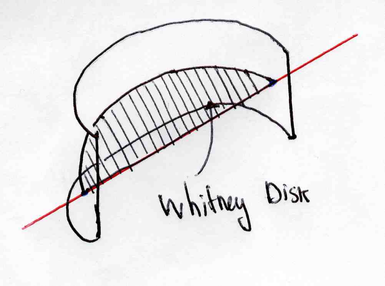
The Whitney Trick

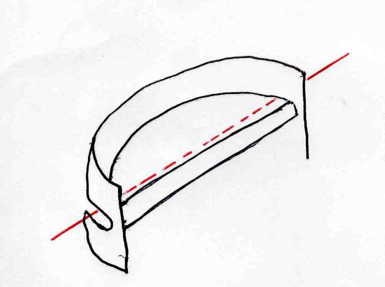
In terms of using the Whitney trick to geometrically realise algebraic intersection numbers, we are presented with the following diagram



By composing a path from P to Q in the attaching sphere of h^k_α with a path from Q to P in the belt sphere of h^{k-1}_β to obtain a loop in W.

We seek to find an embedded disk in the complement of these two spheres whose boundary is this loop. We can then separate the spheres by smoothly pulling one of them along the disc.





A key step is to show that the complement of the two spheres in W is still simply connected, so that any loop in this space bounds an immersed disk, and then to perturb the immersion to an embedding.

Suppose that we label our k-handles $1, 2, ..., \text{rank}(C_k)$ and similarly label our $(k-1)$-handles $(k-1)$-handles $1, 2, ..., \text{rank}(C_{k-1})$. If h_i^k cancels with h_j^{k-1} then $\partial_k(h_i^k) = \pm h_j^{k-1}$.

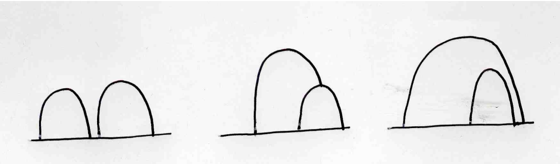
Removing these two handles means that we remove h_i^k, h_j^{k-1} as generators and we also remove the j-th row and the i-th column of the matrix of ∂_k .

Handle creation does the opposite, we add a row and a column which share a common value 1, but take the value 0 otherwise.

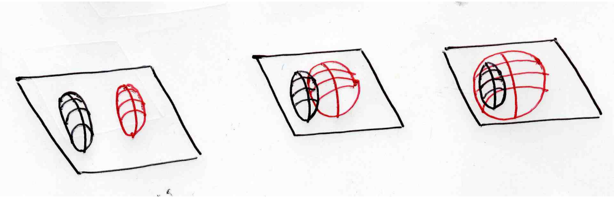
Idea: Given a k-handle, try to create a cancelling pair of $(k+1)$ and $(k+2)$-handles such that the $(k+1)$-handle cancels with the k-handle, that is we trade a k handle for a $(k+2)$-handle.

Theorem: We can trade all 0-handles for 2-handles and all 1-handles for 3-handles.

We can also side a k-handle over an k-handle.



Handle Trading and Sliding



The attaching sphere of the sliding handle travels over a parallel copy of the core of the stationary handle. If h^k_α slides over h^k_β then the boundary map changes as if we replace the basis element h^k_α by $h^k_\alpha \pm h^k_\beta$.

Keeping the basis fixed, this has the effect of changing the matrix of the boundary map $\partial_k : C_k \to C_{k-1}$ by elementary row and column operations.

Dual Handle Decomposition

If $f : W \to [0, 1]$ is a Morse function then so is $-f : W \to [-1, 0]$ where $\text{Crit}(f) = \text{Crit}(-f)$ but a criticial point p of f of index k is now a critical point of $-f$ of index $m + 1 - k$.

This yields a handle decomposition where we replace k-handles by $(m + 1 - k)$-handles. We may then

- Trade all 0-handles and 1-handles for 2 and 3-handles in the handle decomposition of f and flip to obtain a handle decomposition with no m or $(m + 1)$-handles.
- Further trade all 0-handles and 1-handles to obtain a handle decomposition with no $0, 1, m$ or $(m + 1)$-handles.
- Flip one more time to obtain a handle decomposition with no $0, 1, m$ or $(m + 1)$-handles.

We have obtained a handle decomposition with no $0, 1, m$ or $(m+1)$-handles, so the handle chain complex looks like

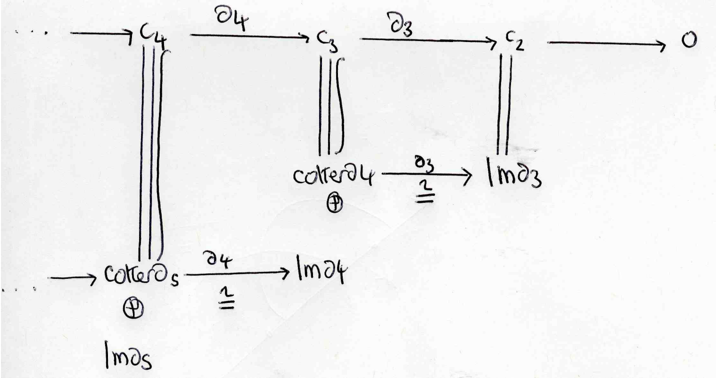
$$0 \to C_{m-1} \xrightarrow{\partial_{m-1}} C_{m-2} \xrightarrow{\partial_{m-2}} .... \to C_3 \xrightarrow{\partial_3} C_2 \to 0$$

The homology of this chain complex coincides with the relative homology groups $H_*(W, M; \mathbb{Z})$. However the fact that $(W : M, M')$ is a h-cobordism and W, M, M' are all simply connected means that the homology $H_*(W, M; \mathbb{Z})$ is zero.

So this chain complex is exact. So ∂_3 is surjective and since its image C_2 is free, ∂_3 splits. This gives a decomposition of C_3 as

$$C_3 \cong \mathrm{Im}\partial_4 \oplus \mathrm{coker}\partial_4 = \ker\partial_3 \oplus \mathrm{coker}\partial_4$$

In particular the restriction $\partial_3 : \text{coker}\partial_4 \to \text{Im}\partial_3$ is an isomorphism. Since the image of $\partial_4 \leq C_3$ is also free abelian, we can play the same game to obtain a diagram



Then each boundary map $\partial_k : C_k \to C_{k-1}$ passes to an isomorphism ∂_k :Coker$\partial_{k+1} \to$Im∂_k. This allows us to diagonalise by elementary row and column operations i.e. by sliding handles, handle creation and handle cancellation. Since these integer matrices are all invertible, the diagonal must consists of 1's and -1's. Reversing orientations if necessary we can assume that all non-zero elements are equal to 1.

This means that all handles are ∂-paired i.e. for any k-handle h_α^k either

- there is a $(k-1)$-handle h_β^{k-1} such that $\partial_k h_\alpha^k = h_\beta^{k-1}$
- there is a $(k+1)$-handle h_γ^{k+1} such that $\partial_{k+1} h_\gamma^{k+1} = h_\alpha^k$

So the set of handles cancel algebraically in pairs, which in turn gives a total geometric cancellation and a handle decomposition with no handles!

The end!