LECTURE 8: SIMPLE HOMOTOPY EQUIVALENCES, WHITEHEAD TORSION & THE S-COBORDISM THEOREM

3/10/11

We saw in Chris Palmer's lecture on the h-cobordism theorem that by sliding, creating and cancelling handles we can eventually make all the boundary maps $\partial_k : C_k \to C_{k-1}$ diagonal with ± 1 down the diagonal, then by reorienting the handles as necessary we can change the -1 diagonal entries to $+1$.

Sliding handles has the effect of changing basis from h^k_α to $h^k_\alpha \pm h^k_\beta$ if we slide h^k_α over h^k_β, or equivalently sliding handles performs elementary row and column operations on the matrix ∂_k, that is, multiplying by elementary matrices $I + E_{ij}$. Here E_{ij} denotes the matrix that is zero everywhere except the i,j^{th} entry which is 1.

Creating cancelling handle pairs has the effect of adding a row and column to the matrix, a 1 in the bottom right entry and zeroes elsewhere, we call this stabilisation by the identity:

$$\partial_k \longmapsto \begin{pmatrix} \partial_k & 0 \\ 0 & 1 \end{pmatrix}$$

Cancelling such a handle pair has the opposite effect: destabilisation.

In the case of an h-cobordism, $\pi_1(M) = \pi_1(W) = \pi_1(M') = 0$ so the incidence numbers $\langle h^{k+1}_\alpha | h^k_\beta \rangle \in \mathbb{Z}\pi_1(W) = \mathbb{Z}$. In general the fundamental group will not be trivial, so ∂_k will have entries in $\mathbb{Z}\pi_1(W)$, and we will have to keep track of the fundamental group.

§1: The Whitehead group & the s-cobordism theorem:

Let R be a ring with identity. Let $GL(n,R)$ denote the general linear group of invertible $n \times n$ matrices. Let

$$GL(R) := \varinjlim_n GL(n,R)$$

be the direct limit given by stabilising by the identity

$$GL(n,R) \subset GL(n+1,R)$$

$$A \longmapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$$

Let $E_n(R) \trianglelefteq GL(n,R)$ be the normal subgroup generated by the elementary matrices $\{I_n + rE_{ij}\}$. Let $E(R) := \varinjlim_n E_n(R)$ be the direct limit.

<u>Lemma</u>: $E(R) = [GL(R), GL(R)]$.

<u>Proof</u>: ($\subset$): Any elementary matrix $I + rE_{ij}$ can be written as the commutator $[I + E_{ik}, I + rE_{kj}]$.

($\supset$): We show that any commutator $[A,B]$ can be written explicitly as a product of elementary matrices:

$[A,B] = ABA^{-1}B^{-1}$ is in $GL(n,R)$ for some sufficiently large n.

$$= \begin{pmatrix} ABA^{-1}B^{-1} & 0 \\ 0 & I_n \end{pmatrix} \text{ in } GL(2n,R)$$

$$= \begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix}\begin{pmatrix} B & 0 \\ 0 & B^{-1} \end{pmatrix}\begin{pmatrix} (BA)^{-1} & 0 \\ 0 & BA \end{pmatrix}$$

We now show any matrix of the form $\begin{pmatrix} C & 0 \\ 0 & C^{-1} \end{pmatrix}$ is a product of elementary matrices:

$$\begin{pmatrix} C & 0 \\ 0 & C^{-1} \end{pmatrix} = \begin{pmatrix} I & C \\ 0 & I \end{pmatrix}\begin{pmatrix} I & 0 \\ -C^{-1} & I \end{pmatrix}\begin{pmatrix} I & C \\ 0 & I \end{pmatrix}\begin{pmatrix} I & -I \\ 0 & I \end{pmatrix}\begin{pmatrix} I & 0 \\ I & I \end{pmatrix}\begin{pmatrix} I & -I \\ 0 & I \end{pmatrix} \quad (*)$$

and

$$\begin{pmatrix} I & X \\ 0 & I \end{pmatrix} = \prod_{i=1}^{n}\prod_{j=1}^{n}(I_{2n} + x_{ij}E_{i,j+n})$$

so $ABA^{-1}B^{-1}$ is a product of elementary matrices. □

<u>Def^n</u>: Let $K_1(R) := GL(R)/E(R)$, the abelianisation of the general linear group.

<u>Remarks</u>: i) $[A] = [B] \in K_1(R) \iff AB^{-1}$ is a product of elementaries.

ii) The group operation for $K_1(R)$ is matrix multiplication, however <u>block addition</u> $A \cdot B := \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ induces the same group operation because $\begin{pmatrix} AB & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} B & 0 \\ 0 & B^{-1} \end{pmatrix}$

and we already saw by (*) that $\begin{pmatrix} B & 0 \\ 0 & B^{-1} \end{pmatrix} = 0 \in K_1(R)$.

iii) The question whether $K_1(R)$ is trivial or not is the same as the question "Does a Euclidean algorithm exist to diagonalise invertible matrices over R?"

Example: Let $R = \mathbb{Z}$. Every integral invertible matrix can be diagonalised to yield a matrix with diagonal entries in ± 1, then using the fact that $\begin{pmatrix} -1 & 0 \\ 0 & (-1)^{-1} \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = 0 \in K_1(R)$ we can get rid of all the minus signs except possibly the top left entry. Thus $K_1(\mathbb{Z}) = \mathbb{Z}_2$.

With the proof of the h-cobordism theorem we saw that by changing the orientation of handles we could make $\partial_k = \begin{pmatrix} \pm 1 & & 0 \\ & \ddots & \\ 0 & & \pm 1 \end{pmatrix}$ into the identity matrix. (So really the class in $\tilde{K}_1(\mathbb{Z}) := K_1(\mathbb{Z}) / \{\pm 1\}$ is all that is important here.)

In general for non-simply connected h-cobordisms we had

$C_k = \mathbb{Z}\pi_1(M)\{k\text{-handles}\}$ with $H_k(C_*) = H_k(\tilde{W}, \tilde{M}; \mathbb{Z}) = 0$,

and by changing the orientation of a handle and the choice of lift to the universal cover we can make

$$\partial_k = \begin{pmatrix} \pm g_1 & & 0 \\ & \ddots & \\ 0 & & \pm g_n \end{pmatrix}$$

into the identity matrix. Thus we are only concerned with the classes of matrices in $K_1(\mathbb{Z}\pi_1) / \{\pm \pi_1\} =: Wh(\pi_1)$ which we call the Whitehead group of π_1.

Examples: i) By the previous example, we see that $Wh(\mathbb{Z}) = 0$.

ii) $Wh(\pi_1(M)) = 0$ for M a surface (either orientable or non).

iii) $Wh(\pi_1(M)) = 0$ for M a compact n-dimensional manifold with universal cover $\widetilde{M} = \mathbb{R}^m$. This is a CONJECTURE, verified in many cases. (The Whitehead group version of the Novikov conjecture.).

Recall: A chain complex (C, d_C) is chain contractible if it is chain homotopy equivalent to the zero chain complex, i.e. $\exists$ chain homotopy $s: C_* \to C_{*+1}$ between 0 and 1:

$$d_C s + s d_C = 1 - 0$$

Lemma: Let C_* be a chain complex of finitely generated projective R-modules. If $H_*(C_*) = 0$, then C is chain contractible.

Proof: By induction on chain length.

Lemma: let C be a chain contractible chain complex with chain contraction s, then

$$d+s: \bigoplus_{*\text{odd}} C_* \xrightarrow{\cong} \bigoplus_{*\text{even}} C_*$$

is a chain isomorphism with inverse $(1+s^2)^{-1}(d+s)$.

Now let $i: \bigoplus_{\text{even}} C_* \xrightarrow{\cong} \bigoplus_{\text{odd}} C_*$ be a chain isomorphism sending bases to bases, then we define the Whitehead torsion of the chain complex C, denoted $\tau(C_*)$, by

$$\tau(C_*) := \tau(i \circ (d+s))$$

where $\tau(i\circ(d+s))$ is the class of the matrix $i\circ(d+s)$ in the Whitehead group.

This is well-defined: • Independence of choice of i: let i' be another choice for i, then $\tau(i\circ(i')^{-1}) = 0$ since $i\circ(i')^{-1}$ is a permutation matrix $\Rightarrow$ $\tau(i\circ(d+s)) = \tau(i\circ(i')^{-1}\circ i'\circ(d+s))$

$$= \tau(i\circ(i')^{-1}) + \tau(i'\circ(d+s))$$

$$= \tau(i'\circ(d+s)).$$

• Independence of choice of chain contraction s is given by lemma 8.9 ii) of Andrew's book.

<u>Def^n</u>: i) The <u>torsion</u> of a chain equivalence $f: C \xrightarrow{\sim} D$ of finite based finitely generated free R-module chain complexes is

$$\tau(f) := \tau(\mathscr{C}(f))$$

where $\mathscr{C}(f)$ is the algebraic mapping cone of f, defined as

$$(d_{\mathscr{C}(f)})_n = \begin{pmatrix} (d_C)_n & 0 \\ (-1)^n f & (d_D)_{n+1} \end{pmatrix} : \mathscr{C}(f)_n = C_n \oplus D_{n+1} \longrightarrow \mathscr{C}(f)_{n-1} = C_{n-1} \oplus D_n .$$

ii) The <u>torsion</u> of a homotopy equivalence $f: X \xrightarrow{\sim} Y$ of finite CW complexes is the torsion of the induced chain equivalence

$$\tilde{f}: C(\tilde{X}) \longrightarrow C(\tilde{Y})$$

ie. $\tau(f) = \tau(\mathscr{C}(\tilde{f})) = \tau(C_*(\tilde{X}, \tilde{Y})) \in Wh(\pi_1(X))$.

iii) We say a homotopy equivalence $f: X \xrightarrow{\sim} Y$ is <u>simple</u> if $\tau(f) = 0 \in Wh(\pi_1(X))$.

Now returning to considering h-cobordisms we have:

<u>Lemma:</u> All ∂_k in $C_*(\tilde{W}, \tilde{M})$ can be diagonalised (with entries in $\pm\pi_1(M)$) $\iff$ $\tau(M \hookrightarrow W) = 0$.

<u>Def^n</u>: An <u>s-cobordism</u> (s for simple) is an h-cobordism $(W; M, M')$ such that $\tau(M \hookrightarrow W) = 0 \in Wh(\pi_1(M))$.

so if $\tau(M \hookrightarrow W) = 0$, we can diagonalise all the ∂_k's, and if $\dim M \geq 5$ we can apply the Whitney trick to cancel all handles geometrically. Thus

<u>S-Cobordism theorem:</u> Let W^{m+1} be an s-cobordism between M^m and N^m, $m \geq 5$. Then W is diffeomorphic to $M \times I$ restricting to the identity $M \subset W \longrightarrow M \times \{0\}$, in particular $M \cong N$.

For $m \geq 5$ all Whitehead torsions can be realised:

<u>Prop^n</u>: Let $m \geq 5$, M a closed m-dimensional manifold with $\pi_1(M) = \pi$. For all $\tau \in Wh(\pi)$, there exists an h-cobordism $(W; M, M')$ such that $\tau(M \hookrightarrow W) = \tau \in Wh(\pi)$.

<u>Proof</u>: c.f. Ranicki 8.22.

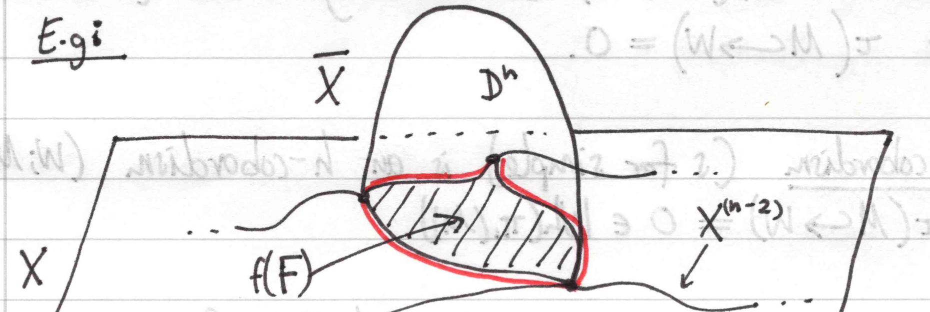
In the following section I will attempt to explain geometrically what it means to be a simple homotopy equivalence.

<u>§2: Simple homotopy equivalences</u>:

In all that follows we will talk about CW complexes, but by thickening cells we might as well be talking about handle decompositions.

<u>Def^n</u>: A finite CW complex $\overline{X}$ is an <u>elementary expansion</u> of a CW complex X if $\overline{X} = X \cup_f D^n$ where D^n is attached to X via <u>part of its boundary</u> $f: F := D^{n-1} \subset \partial D^n \longrightarrow X^{(n-1)}$ such that $f(\partial D^{n-1}) \subset X^{(n-2)}$

<u>E.g.</u>:



equator of $\partial(D^n)$ is attached to $X^{(n-2)}$, lower hemisphere is attached to $X^{(n-1)}$

If $\overline{X}$ is an elementary expansion of X we write $X \overset{e}{\nearrow} \overline{X}$ or $\overline{X} \overset{e}{\searrow} X$. If $X = X_0 \overset{e}{\nearrow} \dots \overset{e}{\nearrow} X_n = Y$ then we write $X \nearrow Y$ or $Y \searrow X$. For a mixture $X = X_0 \nearrow X_1 \searrow \dots \nearrow X_n = Y$ we write $X \nearrow\!\!\searrow Y$ and we say that

X and Y are <u>simple homotopy equivalent.</u>

Wlog if $X \nearrow\searrow Y$ we can always do all expansions first followed by all contractions so $X \nearrow Z \searrow Y$.

<u>Def^n</u>: If (X,Y) is a finite CW pair and $Y \hookrightarrow X$ a homotopy equivalence, then define $\tau(X,Y) := \tau(C_*(\tilde{X},\tilde{Y})) = \tau(\mathcal{C}(\tilde{Y} \hookrightarrow \tilde{X})) \in Wh(\pi_1(Y))$.

<u>Theorem:</u> $X \nearrow\searrow Y$ rel $Y \iff \tau(X,Y) = 0$.

<u>Proof</u>: $(\Rightarrow)$: It is sufficient to show that τ is invariant under elementary expansions rel Y. So suppose $X \overset{e}{\nearrow} X' = X \cup e^{n-1} \cup e^n$, e^n an n-cell, e^{n-1} an $(n-1)$-cell, where $e^{n-1} = \partial D^n - \mathring{F}$. Choose orientations so that

$$\partial[e^n] = [e^{n-1}] + c \qquad \text{some } c \in C_{n-1}(\tilde{X},\tilde{Y}).$$

Since $Y \overset{\sim}{\hookrightarrow} X$ $C_*(\tilde{X},\tilde{Y}) \simeq 0$ chain contractible by earlier lemma. We show we can extend a chain contraction s for $C_*(\tilde{X},\tilde{Y})$ to a chain contraction for $C_*(\tilde{X}',\tilde{Y})$ without changing the Whitehead torsion.

Set $s([e^n]) = 0$, $s([e^{n-1}]) = [e^n] - sc$.

ie
$$\begin{tikzcd}
\cdots \arrow[r, shift left, "\partial"] & C_n \arrow[l, shift left, "s"] \arrow[r, shift left, "\partial"] \arrow[d, phantom, "\oplus"] & C_{n-1} \arrow[l, shift left, "s"] \arrow[r, shift left, "\partial"] \arrow[d, phantom, "\oplus"] & \cdots \arrow[l, shift left, "s"] \\
& \langle e^n \rangle \arrow[ur, "c"] \arrow[r, shift left, "1"] & \langle e^{n-1} \rangle \arrow[l, shift left, "1"] \arrow[ul, "-sc"'] &
\end{tikzcd}$$

This is a chain contraction. For n even, the new $\partial + s$ matrix is

$$\begin{array}{c|cc} & \bigoplus_{\text{even}} C_i & [e^n] \\ \hline \bigoplus_{\text{odd}} C_i & \partial + s & c \\ [e^{n-1}] & 0 & 1 \end{array}$$

for n odd:

$$\begin{array}{c|cc} & \bigoplus_{\text{even}} C_i & [e^{n-1}] \\ \hline \bigoplus_{\text{odd}} C_i & \partial + s & -sc \\ [e^n] & 0 & 1 \end{array}$$

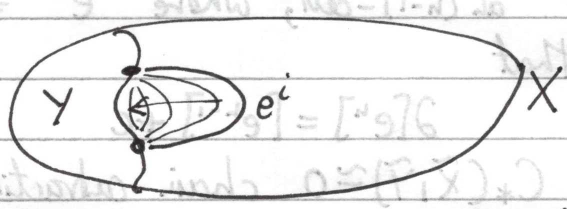
Either can be obtained from $\begin{pmatrix} \partial + s & 0 \\ 0 & 1 \end{pmatrix}$ via elem row operations $\Rightarrow \tau$ unchanged. $\square$

To prove the converse we use the following lemma:

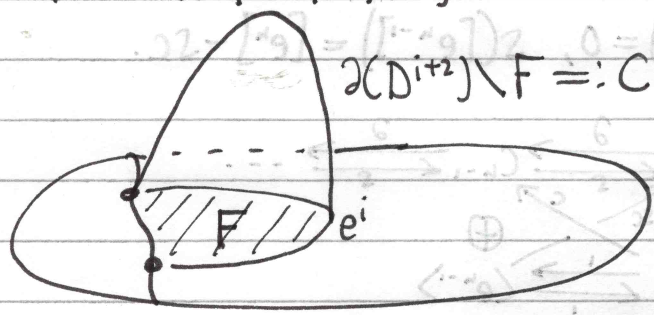
Whitehead cell-trading lemma: Let (X,Y) be a finite CW pair with $\pi_k(X,Y)=0$, $0\leq k\leq n$. Then $\exists\ \overline{X}$, $X \nearrow\searrow \overline{X}$ rel Y such that $\overline{X}-Y$ has no cells of dimension $\leq n$.

Proof: Idea: Any i-cell in $X-Y$, $i\leq n$ can be traded for an $(i+2)$-cell in $\overline{X}-Y$ where $X \overset{e}{\nearrow} X\cup\{e^{i+1}\}\cup\{e^{i+2}\} \overset{e}{\searrow} \overline{X}$.

Let e^i be an i-cell in $X-Y$, $i\leq n$. Since $\pi_i(X,Y)=0$, e^i is homotopic to an i-cell in Y:

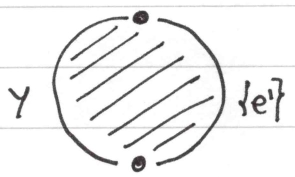


Let $\Phi: (D^i, S^{i-1})\times I \longrightarrow (X,Y)$ be this homotopy. Consider this as a map f from $D^{i+1}\subset \partial D^{i+2} \longrightarrow X$ as we have in an elementary expansion $X' = X\cup\{e^{i+1}\}\cup\{e^{i+2}\}$:

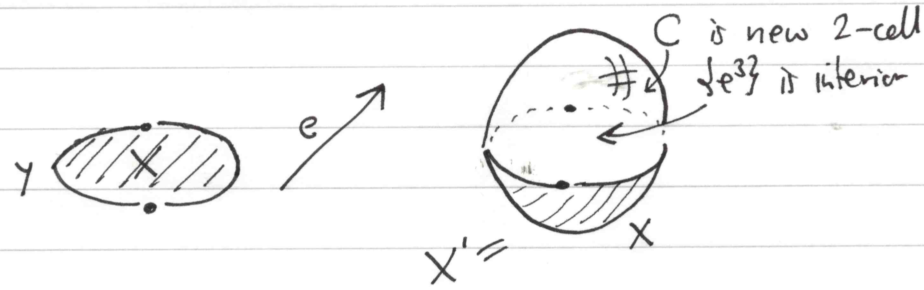


This can also be viewed as an elementary expansion of $\overline{X}$ where $X' = \overline{X}\cup\{e^i\}\cup\{C\}$, for C the $(i-1)$-cell $C=\partial(D^{i+2})\setminus F$. Now collapse C to $F\cap Y$ and $\overline{X} = X'\cup_G Y$ where $G: C\cup Y \longrightarrow Y$ is the elementary collapse. We have thus traded $\{e^i\}$ in $X-Y$ for $\{e^{i+2}\}$ in $\overline{X}-Y$.

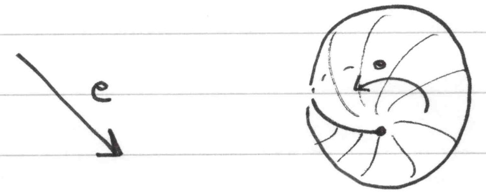
Example: Let $X = D^2$ with the following cell decomposition:



Let $Y = D^1 \hookrightarrow \partial D^2$ be one of the 1-cells & both 0-cells and $\{e^1\}$ the other. We trade $\{e^1\}$ for an $\{e^3\}$.



Now collapse C to identify $\{e^1\}$ with the 1-cell ~~Y⊂~~Y. in Y.



2-cell of X is stretched over C.

We now return to the proof of the theorem: $Y \hookrightarrow X$ with $\tau(X,Y) = 0 \Rightarrow X \searrow Y$ rel Y.

Proof continued: ($\Leftarrow$) Sketch: Since $X \simeq Y$, $\pi_k(X,Y) = 0 \ \forall k$. Since X, Y finite we may apply the cell-trading lemma to trade all cells in $X - Y$ up above $\dim(Y)$, and sufficiently high to be concentrated in dimensions k & $k+1$ ~~& k+2~~ say, where $k = \dim(X)$. So $X \searrow X'$ rel Y where
$X' = Y \cup \{e_i^k\}_{i \in I} \cup \{e_j^{k+1}\}_{j \in J}$.

$$C_*(\tilde{X}, \tilde{Y}): \qquad 0 \longrightarrow C_{k+1}(\tilde{X}, \tilde{Y}) \xrightarrow[\cong]{\partial} C_k(\tilde{X}, \tilde{Y}) \longrightarrow 0$$

Since $\tau(X,Y) = \tau(X',Y)$ by the ($\Rightarrow$) direction of the proof and $\tau(X,Y) = 0$, we may write ∂ as a product of elementary matrices after stabilisation. This gives us a recipe for how to obtain Y for X'

via elementary expansions and contractions (all in $X'-Y$), so this is rel Y. $\square$.

<u>Corollary</u>: Two finite dimensional CW complexes have the same simple homotopy type $\Leftrightarrow$ they have ~~the~~ PL homeomorphic (closed) regular neighbourhoods in some Euclidean space.

<u>Proof:</u> (c.f. Handbook of K-theory vol 1, p. 593)

<u>Idea</u>: • Wlog $X \nearrow\!\!\searrow X'$ is inclusion of one end of a mapping cylinder.

• Take regular neighbourhoods in $\mathbb{R}^n$, $n \geq 6$, of the mapping cylinder to yield a proper h-cobordism between regular neighbourhoods of X and X'. The fact that $X \nearrow\!\!\searrow X'$ means this is an s-cobordism so applying the s-cobordism theorem we get that the regular neighbourhoods are diffeomorphic..

$\square$.