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The Spivak Normal Fibration

0 Intro

Almost in a position to talk about the major successes of surgery. We asked:

(i) When is $M \cong M'$?
(ii) When is M diffeo M'?

h-/s-cobordism changed the question of (ii) to

(iia.) When are M, M' h-/s-cobordant?

STRATEGY: • Show M, M' cobordant via W.
• Surger W to be h-/s-cobordism.

Browder, Novikov, Sullivan and Wall formulated a systematic way of deciding if this can be done. To understand it we need to answer the auxilliary question:

(Q) When is a CW complex htpc to a closed manifold?

THEME: Fibre bundles / Vector bundles
homotopy / topology.

Poincaré Duality

(1) MANIFOLDS HAVE POINCARÉ DUALITY.

⇒ Our CW-complexes had better have it.

<u>Recall</u> For a chain complex of $\mathbb{Z}\pi$ modules associated to our CW complex X:

$$H_n(X;\mathbb{Z}^\omega) := H_n(\mathbb{Z}^\omega \otimes_{\mathbb{Z}\pi} C_*(\tilde{X}))$$

where $\omega : \pi_1(X) \to \mathbb{Z}_2$ our orientation character

<u>Def'n</u> A connected ~~finite~~ m- dimensional geo Poincaré complex (P-space) is a connected CW complex of dim m with an orientation character

$$\omega : \pi_1(X) \to \mathbb{Z}_2$$

and a fundamental class $[X] \in H_m(X;\mathbb{Z}^\omega)$ such that the $\mathbb{Z}\pi$ chain map

$$- \cap [X] : C^{m-*}(\tilde{X}) \longrightarrow C_*(\tilde{X})$$

is a $\mathbb{Z}\pi$-chain equivalence.

<u>Remark</u> • This induces isomorphisms

$$H^{m-*}(\tilde{X};\mathbb{Z}^\omega) \xrightarrow{\sim} H_*(\tilde{X})$$
$$H^{m-*}(\tilde{X}) \xrightarrow{\sim} H_*(\tilde{X};\mathbb{Z}^\omega)$$

• This also works for CW pairs but now $[X] \in H_m(X,\partial X;\mathbb{Z}^\omega)$ and

$$- \cap [X] : H^{m-*}(\tilde{X},\partial\tilde{X};\mathbb{Z}^\omega) \to H_*(\tilde{X}) \text{ etc.}$$

<u>Theorem</u> M^m connected, closed, ~~smooth~~ manifold. There is a finite CW complex X homeomorphic to M s.t. X a P-space.

<u>Remark</u>: If our ~~man~~ orientation character is trivial we return PD of the usual kind. i.e

$\exists\ [X] \in H_m(X)$ s.t.

$$- \cap [X] : H^{m-*}(X) \longrightarrow H_*(X).$$

(+ choose tensor with $\mathbb{Z}\pi$ in chain level)

~~then~~

HAVE WE DEFINED SOMETHING NEW?

<u>Claim</u> There exist P-spaces that are not homotopic to manifolds.

"<u>Proof</u>" (Wall - Poincaré Complexes '67)

For a covering $\tilde{M} \xrightarrow{n:1} M$ of ~~man~~ closed connected 4-manifolds

$$\sigma(\tilde{M}) = n\,\sigma(M)$$

as $\sigma(M) = \frac{1}{3}\int_M p_1(M)$ is a local expression.

Wall produced X_p $\forall p$ prime such that $\pi_1(X_p) = \mathbb{Z}_p$ and

$$\sigma(\tilde{X}_p) \neq p\,\sigma(X_p).$$

□

SO WHEN <u>IS</u> $X \simeq M$??

② Thom Spaces + Spherical Fibrations

<u>Recall</u> A v.b. over a manifold $\xi : M \to BO(k)$

$$E(\xi) \xrightarrow{p} M$$

can have a metric. Define

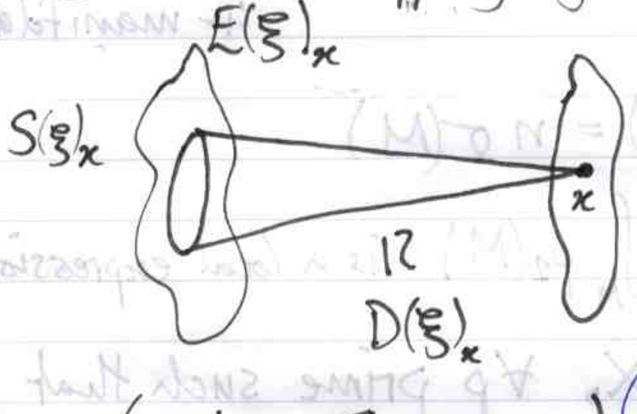
$$D(\xi) = \{ v \in E \mid \|v\| \leq 1 \} \longrightarrow M$$
$$S(\xi) = \{ v \in E \mid \|v\| = 1 \} \longrightarrow M$$

$$\Rightarrow \quad Th(\xi) = \cancel{D}\, D(\xi) \big/ S(\xi)$$

with base point $\infty := S(\xi)/S(\xi)$

<u>Remark</u> • Equivalently, point compactify the fibres of ξ to the same ∞!

• $D(\xi)$ is the mapping cylinder of $S(\xi) \xrightarrow{p_S} M$



• $Th(\xi)$ is the mapping cone of p_S

<u>Theorems</u> (Various Thom isos) (Thom iso twists/untwists)

$$\exists\, U_\xi \in \tilde{H}^k(T(\xi); \mathbb{Z}^w) \cong H^k(D(\xi), S(\xi); \mathbb{Z}^w)$$

s.t. (eg) $U_\xi \cap - : \tilde{H}_*(T(\xi)) \longrightarrow H_{*-k}(D(\xi); \mathbb{Z}^w)$

$$\cong$$

$$H_{*-k}(X; \mathbb{Z}^w)$$

THE RIGHT ANALOGUE OF V.B.s FOR P-SPACES ARE SPHERICAL FIBRATIONS.

<u>Def'n</u> For a fibration

$$S^{k-1} \to E \xrightarrow{p} X$$

Define the disk fibration to be $DE = cyl(p)$.
" ——— " Thom space to be $Th(p) = DE/E$
$= \mathcal{C}(p)$ mapping cone.

<u>Claim</u> (Already shown) Every v.b. has an associated spherical fib

$$S(\xi) \xrightarrow{p_S} M$$

and $Th(\xi) = Th(p_S)$.

<u>Remark</u> • Defining orientation character $\omega: \pi_1(X) \to \mathbb{Z}_2$ as before we obtain the same Thom isomorphism ideas as before. $U_p \in H^k(DE, E; \mathbb{Z}^\omega)$

e.g. $H^*(X) \xrightarrow{i_0^*} H^*(DE) \longrightarrow H^{*+k}(DE, SE; \mathbb{Z})$

(where $i_0 : X \to DE$ the 0-section).

SO THIS IS A MORE BASIC IDEA THAN A V.B. AND CAPTURES MANY OF THE SAME PROPERTIES.

• We can define a "Whitney sum" of two sphere bundles by taking the fibrewise join.

$$S^{k-1} \to E \to X$$
$$S^{\ell-1} \to E' \to X$$
$$S^{k+\ell-1} \longrightarrow E \overset{*}{\cancel{\oplus}} E' \to X$$

in such a way that $S(\xi \oplus \xi') = S(\xi) * S(\xi')$.

<u>Def'n</u> $G(k)$ is the monoid of homotopy equivalences
$$S^{k-1} \xrightarrow{\simeq} S^{k-1}$$

$(k-1)$-spherical bundles are classified by $BG(k)$.

$BG(k) \subset BG(k+1)$ by fibre sum with a triv bundle

$\Rightarrow$ <u>stable</u> classifying space BG.

Reiterating Carmen but IMPORTANT LATER

Construction (Pontrjagin - Thom)

Take a representative for the stable normal bundle $\nu_M : M^m \longrightarrow BO$.

i.e. $i : M \hookrightarrow \mathbb{R}^{m+k}$ s.t. $\nu_i M \oplus TM$ is trivial

Let $(N(M), \partial(M))$ be a tubular n'hood

$\Rightarrow$

$\exists$ diffeo $f : (N(M), \partial N(M)) \xrightarrow{\cong} (D(\nu_i), S(\nu_i))$

The collapse map

$$c : \mathbb{R}^{m+k} \cup \{\infty\} \longrightarrow Th(\nu_i)$$

is f on int $N(M)$ and sends everything else to $\infty = S(\nu)/S(\nu)$.

Claim $c_* : H_{m+k}(\mathbb{R}^{m+k} \cup \{\infty\}) \longrightarrow H_{m+k}(Th(\nu_i)) \cong \mathbb{Z}$.

has $c_*([S^{m+k}])$ a generator.

Proof c is a degree 1 map and is smooth. $\square$

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③ Spivak Normal Fibration

COMPARISON OF OUR 2 CASES:

	spaces	bundles	characteristic classes	classifying spaces
topology	manifolds	v.b.s	pontrjagin	BO
homotopy theory	P-spaces	spherical fibs	Stiefel Whitney	$\pi_*(BO)$ infinite BG $\pi_* BG$ finite

Forgetful maps:

$$\{\text{manifolds}\} \longrightarrow \{\text{P-spaces}\}$$

$$\{\text{v.b.s}\} \longrightarrow \{\text{spherical fibs}\}$$

more about these later

THE ANALOGE OF THE STABLE NORMAL BUNDLE IS THE "SPIVAK NORMAL FIBRATION".

Def'n For X^m a P-space with orientation character $\omega : \pi_1(X) \to \mathbb{Z}_2$, a (k-1)-Spivak normal structure on X is a $(k-1)$ spherical fibration $\nu_X : X \to BG(k)$

$$S^{k-1} \to E(\nu_X) \to X$$

with same orient character as X.

such that we have a pointed map $c : S^{m+k} \longrightarrow Th(\nu_X)$ which agrees with the Thom isomorphism:

$$[X] = U_{\nu_X} \cap c_*[S^{m+k}] \in H_m(X; \mathbb{Z}^\omega)$$

where U_{ν_X} is a choice of Thom class.

Def'n The associated stable fibration

$$\nu_X : X \to BG(k) \to BG$$

is called the Spivak normal fibration

Claim (Sanity check)

The sphere bundle of a representative of ν_M is a $(k-1)$-Spivak normal structure.

Proof $i : M^m \hookrightarrow \mathbb{R}^{m+k} \cup \{\infty\} = S^{m+k}$
$c : S^{m+k} \to Th(\nu M)$
s.t. $c_*[S^{m+k}]$ generates $\tilde{H}_{m+k}(Th(\nu M))$

So a choice of Thom class $U \in \tilde{H}_{\blacksquare k}(Th(\nu M); \mathbb{Z}^{\omega})$ gives

$$\pm[X] = U \cap c_*[S^{m+k}] \in H_m(M; \mathbb{Z}^{\omega})$$

Choose U to give +tive. □

$\boxed{\text{MAIN CLAIM}}$

A CW complex X admits a Spivak normal fibration $\Longleftrightarrow$ X is a P-space.

Discussion of proof:

WHAT CAN WE SAY BEFORE WE START?

X htpc to a finite simplicial cx K (simplic. approx th'm)

A simplicial cx K can be embedded in $\mathbb{R}^{m+k}$ for k large (~~eg~~ Whitney embedding)

WLOG: take $X \hookrightarrow \mathbb{R}^{m+k} \cup \{\infty\}$

with regular neighbourhood $(N(X), \partial N(X))$, then

$$i : X \longrightarrow N(X) \quad (\Rightarrow \text{ BOTH } i_* \text{ and } i^* \text{ are isos})$$

is a homotopy equivalence. Let $[N(X)] \in H_{m+k}(N(X), \partial N(X))$ be a fundamental class agreeing with orient of $\mathbb{R}^{m+k}$.
Note $N(X)$ is oriented. We don't ever need to twist coefficients here. In the sequel we assume <u>everything</u> is orientable and $\pi_1(X) = 0$. (for personal safety reasons).

So $\forall\, u \in H^k(N(X), \partial N(X))$ the following commutes:

$$\begin{tikzcd}
H^{m-*}(X) \arrow[rr, "u \cup (i^*)^{-1}(-)", "\text{``Thom''}"'] \arrow[dr, "\phi"', "\text{``PD''}"] & & H^{m+k-*}(N(X), \partial N(X)) \arrow[dl, "-\cap [N(X)]", "\cong"'] \\
& H_*(X) \cong H_*(N(X)) &
\end{tikzcd}$$

$\square$ $\phi(v) = v \cap (i_*)^{-1}(u \cap [N(X)])$

($\Rightarrow$) If X has an SNF, let u be the Thom class and then this forces ϕ an iso. Take
$[X] = (i_*)^{-1}(U \cap [N(X)]) \Rightarrow X$ a P-space.

What about the converse?

($\Leftarrow$) We want to set the unique u defined by

$$[X] = (i_*^{-1})(u \cap [N(X)])$$

to be the Thom class of some spherical fibration

Now $\quad \partial N(X) \overset{j}{\hookrightarrow} N(X) \overset{i^{-1}}{\simeq} X$ $\qquad$ NOT a fibration

But taking the mapping fibre we can make a fibration

$$p_f : E_f \to X$$

where $\partial N(X) \simeq E_f$. Note that j is a cofibration (i.e. htpy extension) so we extend $\partial N(X) \simeq E_f$ to

$$(N(X), \partial N(X)) \simeq (cyl(p_f), E_f)$$

So as $\quad \mathbb{Z}\langle u \rangle = H^k(N(X), \partial N(X))$
we have $\quad \mathbb{Z}\langle u_{p_f} \rangle = H^k(DE_f, E_f)$ $\quad$ i.e. we have a "Thom class"

But we don't know this has fibre htpic to S^{k-1}.

This is HARD to show.

see e.g. Ranicki - Alg Theory of Surgery II

"$\square$"

④ <u>So What?</u>

<u>Theorem</u> (Browder '62)

A P-space X^m with $\pi_1(X) = 0$ is htpy equiv to a manifold $f: M^m \to X^m$ iff

(1) $\exists$ a v.b. $\xi: X \to BSO(k)$ and $c: S^{m+k} \to Th(\xi)$ s.t.

$$U_\xi \cap c_*[S^{m+k}] = [X]$$

(2) If m even we also need

(i) $m = 4k$: $\sigma(X) = \langle L_k(-\xi), [X] \rangle \in \mathbb{Z}$

(ii) $m = 4k+2$: The $\mathbb{Z}_2$-valued Arf invariant of the self intersection form μ to vanish

$$\mu: \ker\left(f_*: H_{2k+1}(M;\mathbb{Z}_2) \to H_{2k+1}(M;\mathbb{Z}_2)\right) \to \mathbb{Z}_2$$

$\square$

SO THE SOLUTION OF (Q) INVOLVES REDUCING THE SPIVAK NORMAL BUNDLE TO A VECTOR BUNDLE, THEN CHECKING CONDITIONS OF FORMS.

<u>Def'n</u> $J: BO \to BG$ "forgets" the v.b. and takes the spherical fibration.

We want a lift

$$\begin{tikzcd} & BO \arrow[d] \\ X \arrow[r, "\nu_X"] \arrow[ur, dashed] & BG \end{tikzcd}$$

mapping fibre : $G/O \to BO \to BG \to B(G/O)$ so obstruction is $[X, B(G/O)]$

WE CAN NOW MAKE MORE NON-MANIFOLD P-SPACES.

Example (Gitler + Stasheff – The First Exotic Class of BF '65)

Produce a fibration $S^2 \to E \to S^3$

s.t. E has P-space structure but not htpic to a PL-manifold. They use the obstruction to lifting a map from BG to BSO.

THERE IS ALSO SOMETHING VERY DEEP GOING ON HERE WITH THE STIEFEL-WHITNEY CLASSES.

(Adams – ∞-loop spaces) Def'n The ~~i~~ ith Stiefel-Whitney class of a spherical fibration

$$p : E \to X$$

is defined as: $\omega_i(p) := \phi^{-1} Sq^i \phi(1)$
where ϕ are the Thom isos and

$$\begin{tikzcd}
\tilde{H}^{m+k}(Th(p);\mathbb{Z}_2) \arrow[r, "Sq^i"] & \tilde{H}^{m+k+i}(Th(p);\mathbb{Z}_2) \\
H^{m}(X;\mathbb{Z}_2) \arrow[u, "\phi"] & H^{m+i}(X;\mathbb{Z}_2) \arrow[u, "\phi"']
\end{tikzcd}$$

So Stiefel-Whitney classes are invariants of spherical fibrations. In particular, they are htpy invariants where e.g. Pontrjagin classes are not.