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Lecture 13: Quadratic forms

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Aim: Definitions & Examples

Next time: Surgery on forms.

Notation: A will denote a ring with involution $A \to A$, $a \mapsto \bar{a}$

K will be an f.g. A-module

$S(K) = S(K,K) =$ group of sesquilinear pairings $\lambda : K \times K \longrightarrow A$

$S(K) = \mathrm{Hom}_A(K, K^*)$

Definition: ε-Transposition involution

$$T_\varepsilon : S(K) \longrightarrow S(K)$$
$$\lambda \longmapsto T_\varepsilon \lambda = \varepsilon T\lambda$$

where $T\lambda(x,y) = \overline{\lambda(y,x)}$

Def^n: ε-symmetric group $\quad Q^\varepsilon(K) = \ker(1 - T_\varepsilon : S(K) \to S(K))$

ε-quadratic group $\quad Q_\varepsilon(K) = \mathrm{coker}(1 - T_\varepsilon : S(K) \to S(K))$

$\exists$ morphism between: ε-symmetrisation morphism

$$Q_\varepsilon(K) \xrightarrow{1+T_\varepsilon} Q^\varepsilon(K)$$

Ex: For $K = A$, $A \cong S(A)$ so $Q^\varepsilon(A) = \{a \in A \mid \varepsilon\bar{a} = a\}$

$Q_\varepsilon(A) = A / \{b - \varepsilon\bar{b} \in A\}$

Interesting case $A = \mathbb{Z}$, & involution = identity

$$Q^\varepsilon(\mathbb{Z}) = \begin{cases} \mathbb{Z}, & \varepsilon = 1 \\ 0, & \varepsilon = -1 \end{cases}$$

$$Q_\varepsilon(\mathbb{Z}) = \begin{cases} \mathbb{Z}, & \varepsilon = 1 \\ \mathbb{Z}_2, & \varepsilon = -1 \end{cases}$$

for $\varepsilon = +1$, the symmetrisation morphism is $2 : \mathbb{Z} \hookrightarrow \mathbb{Z}$

Quadratic forms are even symmetric forms (also when 2 invertible)

for $\varepsilon = -1$, " $0 : \mathbb{Z}_2 \longrightarrow 0$

for a symmetric form $\exists$ 2 quadratic forms mapping to it, the normal & the Arf.

§ Forms:

Def^n: ε-symmetric form

(K, λ) $\quad \lambda : K \times K \longrightarrow A$

$(x, y) \longmapsto \lambda(x, y) = \varepsilon \overline{\lambda(y, x)}$

We say (K, λ) is non-singular when the adjoint map

$\lambda : K \xrightarrow{\cong} K^*$ is an iso.

$x \longmapsto (y \mapsto \lambda(x, y))$

Example: • Let X^{2n} be a Poincaré complex with oriented cover $(\tilde{X}, \pi, \omega)$ then $H^n(\tilde{X}) \times H^n(\tilde{X}) \to \mathbb{Z}[\pi]$ is a $(-1)^n$-symmetric form over $\mathbb{Z}\pi$ with ω twisted involution

from cup product graded commutative.

• M^{2n}, $\tilde{M}$ univ cover $\quad (\mathrm{im}(\pi_n(M) \longrightarrow H_n(\tilde{M})), \lambda)$ Hom intersection form.

$x \in \pi_n(M)$ can be killed by surgery $\Rightarrow \lambda(x, x) = 0$.

Not a sufficient condition

Example: $M^{2n} = S^n \times S^n$ with $(-1)^n$-symmetric form $(H_n(S^n \times S^n), \lambda)$

$x = (1, 1) \in H_n(S^n \times S^n)$. $\quad \lambda(x, x) = \chi(S^n) = 1 + (-1)^n \in \mathbb{Z}$

so $\lambda(x, x) = 0$ for n odd

Consider the diagonal embedding $\Delta : S^n \hookrightarrow S^n \times S^n$

$\nu_\Delta = \tau_{S^n} : S^n \longrightarrow BO(n)$

This bundle is non-trivial for $n \neq 1, 3, 7$

$\Rightarrow$ Not possible to kill $x = \Delta_*[S^n] \in H_n(S^n \times S^n)$ in dims $n = 1, 3, 7$.

<u>Quadratic forms</u>: Triples (K, λ, μ) where (K, λ) is a symmetric form & $\mu : K \to Q_\varepsilon(A)$ such that for $x, y \in A$

i) $\mu(x+y) - \mu(x) - \mu(y) = \lambda(x, y)$

ii) $\mu(ax) = a\mu(x)\bar{a}$

iii) $\mu(x) + \varepsilon\overline{\mu(x)} = \lambda(x, x)$

(K, λ) non-singular $\Rightarrow$ (K, λ, μ) non-singular.

<u>Example</u>: Consider an n-connected $2n$-dim^l degree 1 normal map $(f, b) : M^{2n} \to X$. This determines a non-singular $(-1)^n$-quadratic form $(K_n(M), \lambda, \mu)$ over $\mathbb{Z}\pi_1(X)$.

$\mu : K_n(M) \to Q_{(-1)^n}(\mathbb{Z}\pi_1(X))$

Interesting thing is if $x \in K_n(M)$ can be killed by surgery then $\lambda(x, x) = 0$, $\mu(x) = 0$. For $n \geq 3$ the converse also holds.

<u>Def^n</u>: Non-singular hyperbolic quadratic forms. Let L be an f.g. projective A-module. $H_\varepsilon(L) = (L \oplus L^*, \lambda, \mu)$ for &

$$\lambda = \begin{pmatrix} 0 & 1 \\ \varepsilon & 0 \end{pmatrix} : L \oplus L^* \to (L \oplus L^*)^* = (L^* \oplus L^{**})$$

$$(x, f) \longmapsto ((y, g) \mapsto f(y) + \varepsilon\overline{g(x)})$$

$$\mu : L \oplus L^* \to Q_\varepsilon(A)$$

check i, ii, iii)

$$(x, f) \longmapsto f(x).$$

for $n \geq 3$ (f, b) is bordant to a h.e $\Leftrightarrow$ $(K_n(M), \lambda, \mu)$ over $\mathbb{Z}\pi_1$ is stably hyperbolic.

i.e. $(K_n(M), \lambda, \mu) \oplus H_{(-1)^n}(F) \cong H_{(-1)^n}(F')$ some F, F' f.g free $\mathbb{Z}\pi_1$ mods.

<u>Split ε-quadratic form</u>: $(K, \psi) \overset{1:1}{\leftrightsquigarrow} (K, \lambda, \mu)$ defined by (K, ψ) where $\psi \in S(K) = \mathrm{Hom}_A(K, K^*)$

$$\lambda(x, y) = \psi(x, y) + \varepsilon\overline{\psi(y, x)}$$

$$\mu(x) = \psi(x, x) \in Q_\varepsilon(A).$$

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Lecture 13 (ctd): Quadratic forms, surgery on forms — Carmen Rovi

A ring w/ involution
K f.g. proj. A-module

(K,λ,μ) it is possible to kill $x \in K \underset{n\geq 3}{\iff} \lambda(x,x)=0$ & $\mu(x)=0$.

Def^n: Orthogonal module $\langle x\rangle^\perp = \{y\in K \mid \lambda(x,y)=0\}$
If x can be killed by surgery $\Rightarrow x \in \langle x\rangle^\perp \Rightarrow \langle x\rangle \subseteq \langle x\rangle^\perp$

$$(K,\lambda,\mu) \xrightarrow{\text{effect of surgery}} (K',\lambda',\mu') = (\langle x\rangle^\perp/\langle x\rangle, \lambda',\mu')$$

More generally: $L \subseteq K$ with $i: L \hookrightarrow K$
$$L^\perp = \{y \in K \mid \lambda(x,y)\ \forall x\in L\} = \ker(i^*\lambda : K \to L^*)$$

can compose
$$\begin{tikzcd} K \arrow[r, "\lambda"] \arrow[rr, bend right, "i^*\lambda"'] & K^* \arrow[r, "i^*"] & L^* \end{tikzcd}$$

Def^n: (K,λ,μ) is non-singular hyperbolic if $\exists L$ s.t. L is a direct summand of K, and it fits into a SES:

$$0 \to L \xrightarrow{i} K \xrightarrow{i^*\lambda} L^* \to 0$$

Def^n: A sublagrangian of (K,λ,μ) is a direct summand of $L \subseteq K$ s.t
$\mu(L) = \{0\} \in Q_\varepsilon(A)$
$\lambda(L\times L) = \{0\}$ ie $L \subseteq L^\perp$

$$(K,\lambda,\mu) \xrightarrow{\text{kill } L} (L^\perp/L, \lambda',\mu')$$

Def^n: A lagrangian of (K,λ,μ) is a sublagrangian s.t $L^\perp = L$ ie s.t effect of surgery is $(0,0,0)$.

Main result: The inclusion of a sublagrangian into a non-singular quadratic form
$$i:(L,0,0) \to (K,\lambda,\mu)$$
extends to an isomorphism
$$f: \underbrace{H_\varepsilon(L)}_{(L\oplus L^*,\lambda,\mu)} \oplus \underbrace{(L^\perp/L,\lambda',\mu')}_{\text{effect of killing } L} \xrightarrow{\cong} (K,\lambda,\mu).$$

Example: let $(f,b): M\to X$ be an n-connected $2n$-dim^l deg 1 normal map. This determines a nonsingular $(-1)^\varepsilon$-quadratic form

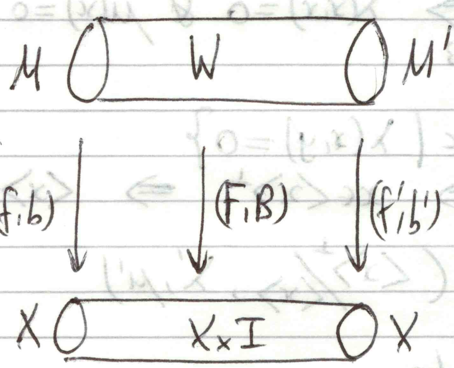
$$(K_n(M),\lambda,\mu) \text{ over } \mathbb{Z}[\pi_1(X)]$$

If $x \in K_n(M)$ can be killed, then $\lambda(x,x)=0$, $\mu(x)=0$ & if $n\geq 3$ converse holds.

$x \in K_n(M)$ generates a submodule $\langle x \rangle = L \subset K_n(M)$

$\Updownarrow$

x can be killed on the embedding $S^n \times D^n \hookrightarrow M$.



The trace is an n-connected normal bordism
$$(W^{2n+1}; M^{2n}, M'^{2n}) \longrightarrow X \times (I; \{0\}, \{1\})$$
s.t $K_n(W, M) = 0$

Quadratic form after surgery $\quad (K_n(M'), \lambda', \mu') = (L^{\perp}/L, \lambda', \mu')$

$$L = \mathrm{im}(K_{n+1}(W, M) \longrightarrow K_n(M))$$

Before we had $\quad (K, \lambda, \mu) \xleftrightarrow{1:1} (K, \psi)$

Def^n: Sublagrangian of (K, ψ) is $(i, \theta) : (L, 0) \longrightarrow (K, \psi)$

Def^n: Lagrangian is a sublagrangian s.t the following sequence is exact

$$0 \longrightarrow L \xrightarrow{i} K \xrightarrow{i^*(\psi + \epsilon\psi^*)} L^* \longrightarrow 0 \qquad (\text{since } L^{\perp} = L)$$

Theorem: The inclusion $(i, \theta) : (L, 0) \to (K, \psi)$ extends to an iso

$$(f, \chi) : H_\epsilon(L) \xrightarrow{\cong} (K, \psi)$$

Proof: Suppose (K, ψ) has a Lagrangian then $(i, \theta) : (L, 0) \longrightarrow (K, \psi)$

$$\Downarrow$$
$$(f, \chi) : H_\epsilon(L) \xrightarrow{\cong} (K, \psi)$$

determines a Lagrangian $f(L^*) \subset K$.

1) Choose a Lagrangian (K, ψ) complementary to L, $i : L \to K$

recall $\quad 0 \to L \xrightarrow{i} K \underset{j'}{\overset{i^*(\psi+\epsilon\psi^*)}{\rightleftarrows}} L^* \to 0 \quad$ is exact

we ~~exhibit~~ choose a splitting j'

$j' : L^* \longrightarrow K$ is <u>not</u> in general the inclusion of a Lagrangian

2) choose $j = j' + ik : L^* \longrightarrow K$ a new splitting s.t.

$$L^* \xrightarrow{j^*\psi j} L$$

$$j^*\psi j = (j'+ik)^*\psi(j'+ik) = \cdots = j'^*\psi j' + \underset{: L^* \to L}{\text{map}} \qquad \in Q_\varepsilon(L^*)$$

3) Choose k to be $-j'^*\psi j' : L^* \longrightarrow L$

$$\Rightarrow \quad j^*\psi j = 0 \in Q_\varepsilon(L^*)$$

Then the corresponding $j : L^* \longrightarrow K$ is now the inclusion of a Lagrangian so extends to a map

$$H_\varepsilon(L) \xrightarrow[\sim]{} (K, \psi). \qquad \square.$$

(i) Cobordism is an equivalence relation on ε-quadratic forms (K, λ, μ) over A.

(ii) (K, λ, μ) and (K', λ', μ') are cobordant if the following iso. exists:

$$(K, \lambda, \mu) \oplus H_\varepsilon(F) \cong (K', \lambda', \mu') \oplus H_\varepsilon(F')$$