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The Even-Dimensional Surgery Obstruction

31/01/12

⓪ Intro

We have seen that given a deg 1 normal map, ~~that~~ by a finite sequence of surgeries we can make the map n-connected, where $2n \leq m$ = dimension of the map.

$\therefore$ WLOG we start with

$$(f,b): M^m \longrightarrow X^m$$

n-connected deg 1 normal map for $m = 2n$ or $2n+1$.

We now wish to do surgery to kill elements $x \in \pi_{n+1}(f)$. We know that IF we can kill $x \in \pi_{k+1}(f)$ with effect M', we have:

$$\begin{tikzcd}
K_{k+1}(W,M) \arrow[rr, bend left, "x"] \arrow[dr] & & K_k(M) \arrow[rr, bend left, "x!"] \arrow[dr] & & K_k(W,M) \\
& K_{k+1}(W,\partial W) \arrow[ur] \arrow[dr] & & K_k(W) \arrow[ur] \arrow[dr] & \\
K_{k+1}(W,M') \arrow[ur] \arrow[rr, bend right, "x'"'] & & K_k(M') \arrow[ur] \arrow[rr, bend right, "(x')!"'] & & K_k(W,M)
\end{tikzcd}$$

where if $k = n$:

$$x : \mathbb{Z}[\pi] \longrightarrow K_n(M)$$
$$1 \longmapsto x$$

$$x^! : \quad y \longmapsto \lambda(x,y)$$

We will consider the case $m = 2n$ and show the following:

(1) The obstruction to making (f, b) $(n+1)$-connected is weaker than $\mu(x) = 0$ $\forall x \in \pi_{n+1}(f)$. It is that $K_n(M)$ is "stably isomorphic" to a sum of hyperbolics.

(2) This obstruction lives in the group $Q_{(-1)^n}(\mathbb{Z}[\pi])$ considered up to algebraic cobordism and only on f.g. free-modules:

$$\sigma_*(f, b) \in L_{2n}(\mathbb{Z}[\pi])$$ "even dim surgery obstruction grp"

(sometimes written $L_{2n}(\pi)$ where the group ring is understood). It is the ONLY obstruction

We will calculate $L_{2n}(\mathbb{Z}[\pi])$ for simple cases and use it to make examples.

If there is time, I want to talk about algebraic L-theory and Ranicki's "instant surgery obstruction".

1 Surgery Obstruction

Let $m = 2n$ and $n \geq 3$.

We have seen:

(i) $x \in \pi_{n+1}(f)$ can be killed by n-surgery on (f,b)

$$\Longleftrightarrow \quad \mu(x) = 0 \quad \text{(Wall's embedding th'm)}$$

(ii) A deg 1 normal bordism W has $K_*(W,M) = 0$, $* \neq n+1$

$\Longleftrightarrow$ W is the trace of ℓ n-surgeries killing $x_1, x_2, \dots, x_\ell \in \pi_{n+1}(f)$

In this case $K_{n+1}(W,M) = \mathbb{Z}\pi^\ell$.

We now need to worry about destroying the connectivity of (f,b) (recall we eventually intend to use Whitehead's th'm). Consider our braid:

$$\begin{tikzcd}[column sep=small]
K_{n+1}(M') \arrow[dr, "3\ \cong"'] \arrow[rr, bend left] & & K_{n+1}(W,M) \arrow[dr] \arrow[rr, bend left, "x"] & & K_n(M)\ \text{2} \arrow[dr] \arrow[rr, bend left, "x^!"] & & K_n(W,M') \arrow[dr] \arrow[rr, bend left] & & K_{n-1}(M') \\
& K_{n+1}(W) \arrow[ur] \arrow[dr] & \mathbb{Z}[\pi]^\ell \arrow[u, equal] & H_n(W,\partial W) \arrow[ur] \arrow[dr] & & K_n(W) \arrow[ur] \arrow[dr] & \mathbb{Z}\pi^\ell \arrow[u, equal] & K_{n-1}(W,\partial W) \arrow[ur, "\cong\ 1"'] \arrow[dr] & \\
K_{n+1}(M) \arrow[ur] \arrow[rr, bend right] \arrow[d, equal] & & K_{n+1}(W,M') \arrow[ur] \arrow[rr, bend right] \arrow[d, equal] & & K_n(M')\ \text{2} \arrow[ur] \arrow[rr, bend right] & & K_n(W,M) \arrow[ur] \arrow[rr, bend right] \arrow[d, equal] & & K_{n-1}(M) \arrow[d, equal] \\
0 & & 0 & & & & 0 & & 0
\end{tikzcd}$$

In general, (f',b') will only be $(n-1)$-connected as, by the fact that homology of the top row is the homology of the bottom row of a braid:

$$K_{n-1}(M') = \operatorname{coker} x^! \quad \text{1}$$
$$K_n(M') = \ker x^! / \operatorname{im} x \quad \text{2}$$
$$K_{n+1}(M') = \ker x \quad \text{3}$$

(and other kernels unchanged).

We want to make all of these vanish.

(iii) $\langle x_1, x_2, \dots, x_\ell \rangle = \langle x \rangle =: L$ is a sublagrangian of $(K_n(M), \lambda, \mu)$

$$\Leftrightarrow K_{n-1}(M') = K_{n+1}(M) = 0.$$

(iv) If L is a sublagrangian then L is a Lagrangian

$$\Leftrightarrow K_n(M') = 0$$

$\Leftrightarrow f'$ a htpy equivalence

So Wall's embedding theorem is not enough. To control the connectivity of the effect we require an algebraic condition on $\langle x \rangle \subset K_n(M)$.

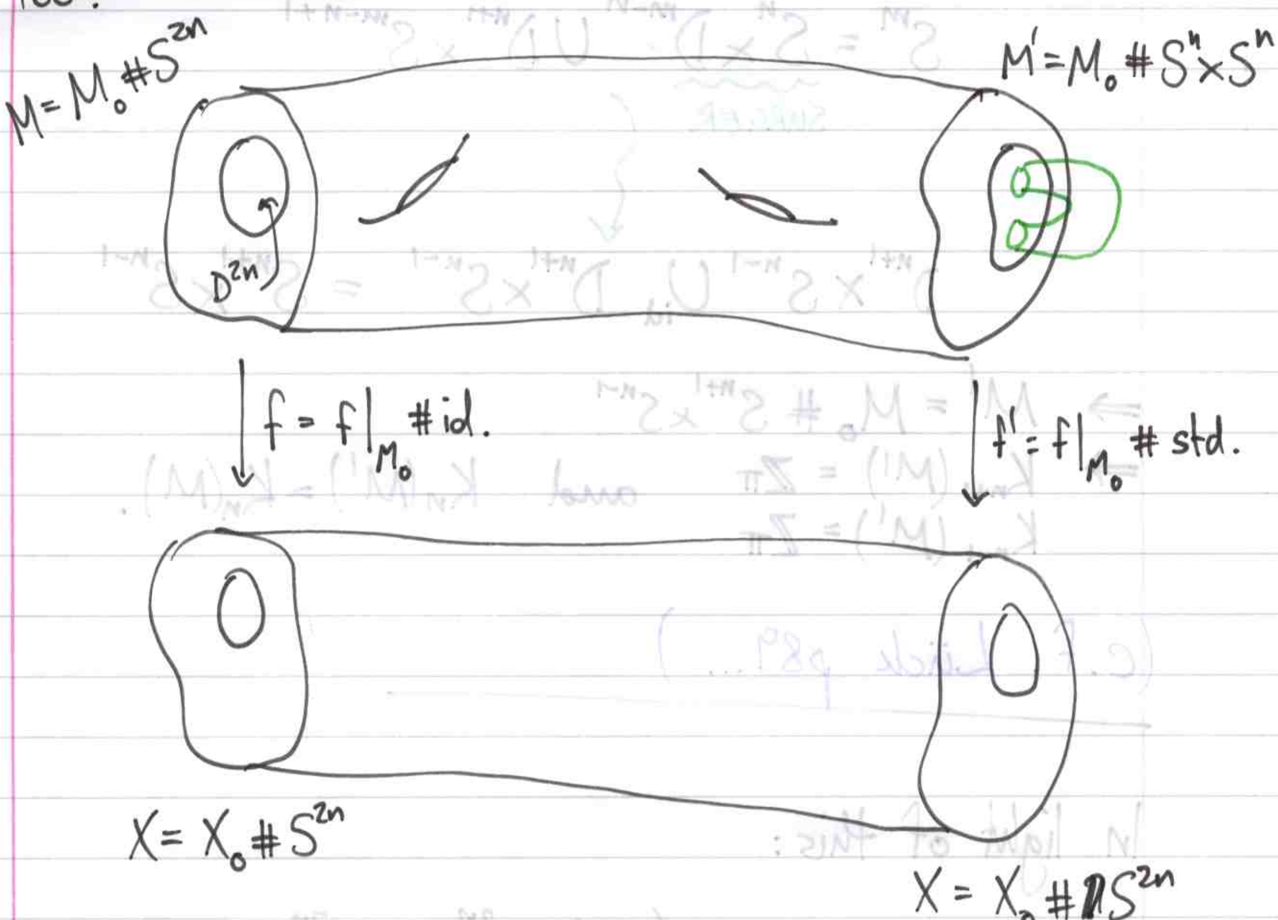
Examples We can always kill $0 \in K_i(M)$, for any i. It is interesting to consider $i = n, n-1$.

$\boxed{i = n-1}$ Looking at

$$\begin{array}{ccc} S^{n-1} & \xrightarrow{g} & M \\ \downarrow & & \downarrow f \\ D^{n} & \xrightarrow{h} & X \end{array} = \phi$$

$\phi = 0 \Rightarrow g$ htpically trivial $\Rightarrow$ take it to happen inside a small disk.

By Poincaré disk th'm can take D^n to be in a small disk too:



<u>TO SEE THIS</u>

As the surgery happens inside S^{2n} it reduces to the standard example

$$S^m = \partial(D^{n+1} \times D^{m-n}) = S^n \times D^{m-n} \cup D^{n+1} \times \underbrace{S^{m-n-1}}_{\text{SURGER THIS}}$$

$$\Big\downarrow \text{Surgery}$$

$$S^n \times D^n \cup_{id} S^n \times S^n = S^n \times S^n$$

the surgery kernel module of $S^n \times S^n \to S^{2n}$

$\Rightarrow K_{n+1}(M') = K_{n-1}(M') = 0$

and $K_n(M') = K_n(M_0 \# S^n \times S^n) = K_n(M) \oplus H_{(-1)^n}(\mathbb{Z}\pi)$

$\boxed{i=n}$ Same again!

$$S^m = \underbrace{S^n \times D^{m-n}}_{\text{SURGER}} \cup D^{n+1} \times S^{m-n+1}$$

$$\downarrow$$

$$D^{n+1} \times S^{n-1} \cup_{id} D^{n+1} \times S^{n-1} = S^{n+1} \times S^{n-1}$$

$\Rightarrow M' = M_0 \# S^{n+1} \times S^{n-1}$

$\Rightarrow K_{n+1}(M') = \mathbb{Z}\pi$ and $K_n(M') = K_n(M)$.
$K_{n-1}(M') = \mathbb{Z}\pi$

(c.f. Lück p89...)

In light of this:

__Theorem__ Suppose $(f,b): M^{2n} \longrightarrow X^{2n}$ an n-conn, deg 1 normal map and $\exists\, k, \ell$ and an iso of forms:

$$(K_n(M), \lambda, \mu) \oplus H_{(-1)^n}(\mathbb{Z}\pi^k) \cong H_{(-1)^n}(\mathbb{Z}\pi^\ell) \quad (*)$$

then (f,b) is deg 1 normal bordant to a htpy equivalence.

__Proof__ Introduce the $H_{(-1)^n}(\mathbb{Z}\pi^k)$ by performing k $(n-1)$-surgeries on $0 \in \pi_n(f)$. This gives $(f', b'): M' \longrightarrow X$ n-conn with

$$(K_n(M'), \lambda', \mu') \cong H(\mathbb{Z}\pi^\ell)$$

perform ℓ n-surgeries on $x_1, \dots, x_\ell$ generating the lagrangian. $\square$

<u>Example</u> I will explain why a non-trivial example of this is difficult.

What would an example require?

- An n-connected map $(f,b): M^{2n} \longrightarrow X^{2n}$

  For simplicity's sake we might try to take X to be a homology sphere so from the LES
  $$0 \longrightarrow \pi_{k+1}(f) \xrightarrow{\cong} \pi_k(M) \longrightarrow 0$$

  and hence $K_n(M) \cong H_n(M)$. So we would want...

- A manifold M^{2n} with homology vanishing outside the middle, top and bottom dimensions.

- An intersection pairing on $H_n(M) = K$ s.t.

$$(K, \lambda, \mu) \oplus H_{(-1)^n}(\mathbb{Z}^k) \cong H_{(-1)^n}(\mathbb{Z}^\ell)$$

TRIVIAL EXAMPLES:

$M = \Sigma^{2n}$ a homology sphere
$M = \#_k S^n \times S^n$

$M = \Sigma^{2n} \# \#_k S^n \times S^n$

BUT there are no forms (K, λ, μ) of this form over $\mathbb{Z}$ that are not hyperbolic already!

We will return to this later.

ASIDE (Lengthy detour to remember the original motivation)

Recall one of original questions was

(Q) If $N \simeq N'$ htpc, is $N \cong_{diff} N'$?

Taking some results about manifolds with boundary on faith, we can now partially answer this!

(A) Let $g : N^{2n-1} \xrightarrow{\simeq} (N')^{2n-1}$ for $n \geq 3$.

g determines a trace CW-cx $(X, \partial X)$ that is a geometric Poincaré cx with $\partial X = N \sqcup \overline{N}'$.

We want to improve $(X, \partial X)$ to be a manifold because then it is an h-cobordism we would have $N \cong_{diff} N'$.

We need:

(i) A vector bundle reduction of the Spivak normal fibration

$$\nu_X : X/\partial X \longrightarrow BG \longrightarrow B(G/O)$$

triv

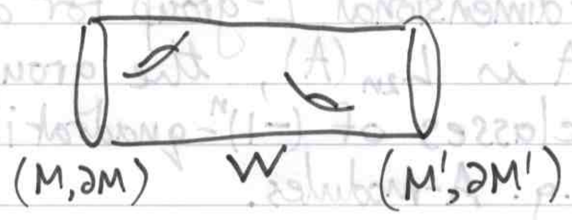
EQUIV: A deg 1 normal map $(f,b) : (M, \partial M) \to (X, \partial X)$ which is already a htpy equiv on ∂.

We can do surgery to make (f,b) n-connected. Hence we also need:

(ii) $\exists\, k, \ell$ s.t.

$$(K_n(M,\partial M),\lambda,\mu) \oplus H_{(-1)^n}(\mathbb{Z}\pi^k) \cong H_{(-1)^n}(\mathbb{Z}\pi^\ell)$$

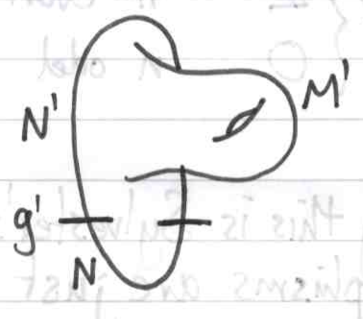
In which case we can surger (f,b) to a htpy equiv rel ∂.
We have "improved f to an h-cobordism".



W a manifold with corners.

$\Downarrow$

$\partial M' = N \sqcup \bar{N}'$



an h-cobordism.

② L-Theory

To calculate the surgery obstruction and show we really have found the whole obstruction we need a better algebraic setting.

Def'n The even-dimensional L-group for a ring with involution A is $L_{2n}(A)$, the group of algebraic cobordism classes of $(-1)^n$-quadratic forms on free, f.g. A-modules.

Examples

$\boxed{A = \mathbb{R}}$

$$L_{2n}(\mathbb{R}) \cong \begin{cases} \mathbb{Z} & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$$

Proof Essentially this is Sylvester's Law of Inertia. The isomorphisms are just the signature. □

$\boxed{A = \mathbb{Z}}$

$$L_{2n}(\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & n \text{ even} \\ \mathbb{Z}_2 & n \text{ odd} \end{cases}$$

I will say slightly more than Andrew's book but won't prove it all.

Proof The symmetrisation map

$$1+T = 2 : Q_+(\mathbb{Z}) \longrightarrow Q^+(\mathbb{Z}) = \mathbb{Z}$$

is an injection. Hence (+)-quadratic forms over $\mathbb{Z}$ correspond to even elements of $Q^+(\mathbb{Z}) = \mathbb{Z}$. These are alternatively characterised as λ s.t. $\lambda(x,x) \equiv 0 \pmod 2$ $\forall x$.

Claim 1 $\sigma(\lambda) = 0$ iff $(K,\lambda) \cong \bigoplus_k H$

Claim 2 $\sigma(\lambda)$ divisible by 8.

For Claim 1:

Let $\underset{\neq}{\overset{0}{x}} \in (K)$ s.t. $\lambda(x,x)=0$. WLOG x is indivisible. i.e. $x = \beta x'$ for $\beta \in \mathbb{Z}$ $\Rightarrow \beta = \pm 1$

The existence of such an x is highly non-trivial and requires some algebraic geometry. See Milnor-Husemoller Lemma 4.1, "Meyer's th'm" 3.2, and Theorem 2.2.

λ is nonsingular over $\mathbb{Z} \Rightarrow \exists\, y \in K$ s.t. $\lambda(x,y)=1$
Now $\lambda(y,y) = 2a$, some $a \in \mathbb{Z}$. $\Rightarrow$ take basis $\{x, y-ax\}$ for $\mathrm{span}\langle x,y\rangle$.

$\Rightarrow K = \mathrm{span}\langle x,y\rangle \oplus \mathrm{span}\langle x,y\rangle^{\perp}$

with $\lambda|_{\langle x,y\rangle} = \begin{pmatrix} 0 & 1 \\ (-1)^n & 0 \end{pmatrix}$.

Repeat. □

For Claim 2:

Define a <u>characteristic element</u> $x \in K$ to be an element s.t.

$$\lambda(x,y) \equiv \lambda(y,y) \pmod 2 \quad \forall y \in K.$$

$x \in (K,\lambda)$ characteristic $\Rightarrow$ $x+e_+ + e_- \in (K\oplus\mathbb{Z}\oplus\mathbb{Z}, \overbrace{\lambda\oplus\langle 1\rangle\oplus\langle -1\rangle}^{\lambda'})$ characteristic as

$$\begin{aligned}
&\lambda'(x+e_+ + e_-, \{y+a+b\}) \\
&= \lambda(x,y) + a - b \\
&\equiv \lambda(y,y) + a - b \pmod 2 \\
&\equiv \lambda(y,y) + a^2 - b^2 \pmod 2 \quad \left(\text{as } a^2 \equiv a\right) \\
&\equiv \lambda'(y+a+b, y+a+b) \pmod 2
\end{aligned}$$

<u>Theorem</u> Indefinite nonsingular forms λ_1, λ_2 are equivalent iff they have same rank signature + parity.

<u>Proof</u> Not much more work to show this. See Theorem 1.2.14 Gompf-Stipsicz. □

$$\Rightarrow (K \oplus \mathbb{Z} \oplus \mathbb{Z}, \lambda') \sim \left(\mathbb{Z}^{b_2^+ +1} \oplus \mathbb{Z}^{b_2^- +1}, (b_2^+ +1)\langle 1 \rangle \oplus (b_2^- +1)\langle -1 \rangle\right)$$

and in this new basis, every characteristic element has odd components i.e. $x + e_1 + e_2 \sim (w, z)$ for w, z odd. Now an odd number is s.t.

$$w^2 \equiv 1 \pmod 8.$$

Hence:

$$\begin{aligned}
\lambda(x,x) &= \lambda(x,x) + 1 - 1 \\
&= \lambda'(x+e_1+e_2, x+e_1+e_2) \\
&= w^2(b_2^+ + 1) - z^2(b_2^- + 1) \\
&\equiv b_2^+ - b_2^- \pmod 8 \\
&\equiv \sigma(\lambda) \pmod 8.
\end{aligned}$$

But (K, λ) even $\Rightarrow 0$ is characteristic $\Rightarrow \sigma(\lambda) \equiv 0 \pmod 8$. $\square$

So we have a well defined homo:

$$\sigma : L_0(\mathbb{Z}) \longrightarrow 8\mathbb{Z} \subset \mathbb{Z}.$$

The E_8 form has $\sigma(E_8) = 8 \Rightarrow \sigma/8$ is an iso.

Now (-1)-quadratic forms!

Firstly via the homo $\mathbb{Z} \longrightarrow \mathbb{Z}_2$ it is equivalent to consider forms over $\mathbb{Z}_2$. This is partly justified by this.

Claim 2 split quadratic forms over $\mathbb{Z}\oplus\mathbb{Z}$

$$\psi : (a,b) \mapsto \begin{pmatrix} a & 1 \\ 0 & b \end{pmatrix} \qquad \psi' : (a,b) \longmapsto \begin{pmatrix} a+2i & 1 \\ 0 & b+2j \end{pmatrix}$$

are equiv $\forall i,j$.

Proof $\psi \sim \psi'$ iff $\exists\ \chi : K \to K^*$ s.t. $\psi - \psi' = \chi - \varepsilon\chi^*$

Take $\chi = \begin{pmatrix} i & 0 \\ 0 & j \end{pmatrix}$. $\square$

Similarly for larger modules.

$$\Rightarrow \quad Q_{(-1)}(\mathbb{Z}) \longrightarrow Q_{(-1)}(\mathbb{Z}_2) \quad \text{an iso}.$$

Restrict to $\mathbb{Z}_2$ (note sym and anti-sym now same)

Now we can always choose a symplectic basis for $(\mathbb{Z}_2^{2m}, \lambda, \mu)$ call it $\{x_1, y_1, x_2, y_2, \dots, x_m, y_m\}$

Def'n $\operatorname{Arf}(K,\lambda,\mu) := \sum \mu(x_i)\mu(y_i)$

Consider $(\mathbb{Z}_2 \oplus \mathbb{Z}_2, \lambda)$ where

$$\lambda((a,b),(c,d)) = ad - bc$$

The Arf invariant captures the information that there are precisely 2 quadratic extension to this form

- $\mu_1((a,b)) := a + b + bc$
- $\mu_2((a,b)) := ab$

characterised by
- $\mu_1(x_1) = \mu_1(y_1) = 1$
- $\mu_2(x_1) = \mu_2(y_1) = 0$

Show for yourself these give the same sym form

Note μ_1 is not equiv to μ_2 and μ_2 is hyperbolic.

The classifications follows from this and noting that

$$\mu_1 \oplus \mu_1 = \mu_2 \oplus \mu_2$$

i.e. Arf measures whether we have an odd or even # of copies of μ_1 in a form.

$\square$

<u>Theorem</u> The surgery obstruction of an n-conn $2n$-dim, deg 1, normal map $(f,b): M^{2n} \longrightarrow X^{2n}$ is the class

$$\sigma_*(f,b) := [(K_n(M), \lambda, \mu)] \in L_{2n}(\mathbb{Z}\pi, \mathfrak{z}(X))$$

~~an is a normal bordism~~ and vanishes iff (f,b) is normal bordant to a htpy equiv.

<u>Proof</u> ~~Well defined:~~ We have seen ($\Rightarrow$) already.

($\Leftarrow$) We show ~~$\sigma_*(f,b) = \sigma_*(f',b')$~~

$(f,b), (f',b')$ deg 1 normal bordant $\Rightarrow$ $\sigma_*(f,b) = \sigma_*(f',b')$.

Example

Recall my attempts from earlier. We can now see that over $\mathbb{Z}$, if • $(K,\lambda) \oplus kH \cong \ell H$ then (K,λ) hyp

• $(K,\lambda,\mu) \oplus kH \cong \ell H$ " "

as Arf + signature unaffected by stabilisation.

For $(f,b): M^{2n} \to X^{2n}$ n-conn deg 1 normal and K_n f.g. free to make it in L_{2n}

Note that K_n is only stably free ~~but we shall see that~~ in general.

<u>Proposition</u> The class $[(K_n, \lambda, \mu)] \in L_{2n}(\mathbb{Z}\pi_1(X))$ is a deg 1 normal bordism invariant.

<u>Proof</u> Let (f,b), (f',b') be deg 1 normal bordant via W

Consider $\quad (F,B):(W,\partial W) \longrightarrow (X, \partial X)$ a $2n+1$ dim, deg 1 normal map

With surgeries on the interior of W, this relative ∂ map is rel ∂ bordant to an n-connected bordism (with same boundary). So there exists an n-conn cobord between (f,b), (f',b'). WLOG ~~W is n-conn~~. (F,B) is n-conn ~~as well~~.

Consider the LES

$$\longrightarrow K_{i+1}(W,M) \longrightarrow K_i(M) \longrightarrow K_i(W) \longrightarrow$$

Connectivity of M, W give

$$0 \to K_{n+1}(W) \to K_{n+1}(W,M) \to K_n(M) \to K_n(W) \to K_n(W,M) \to 0$$

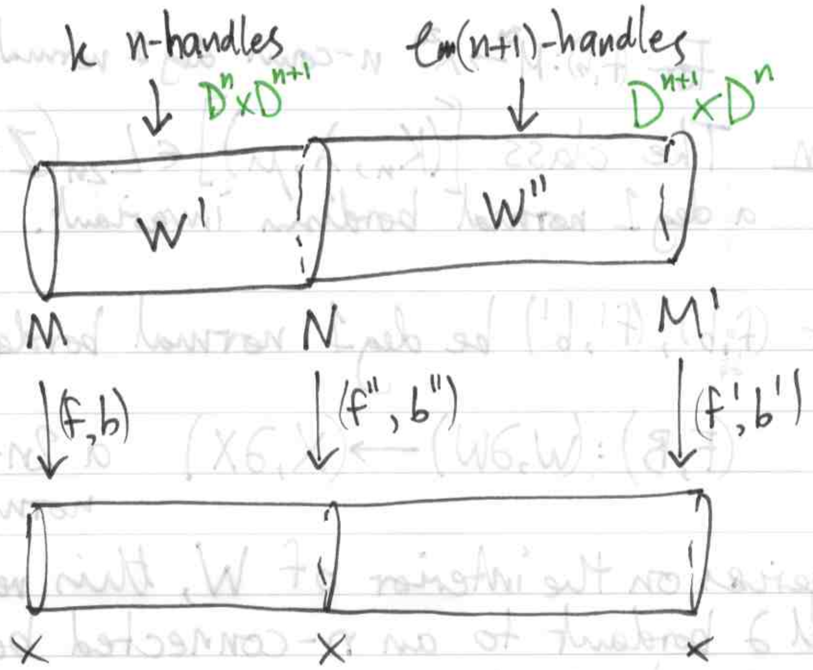
and as $H_i(X \times I, X) = 0 \quad i \neq 0$ we have LES for (F,f)

$$\ldots \to 0 \to K_i(W,M) \xrightarrow{\cong} H_i(W,M) \to 0 \to \ldots$$

$$\Rightarrow \quad H_i(W,M) = 0 \quad \text{for } i \neq n, n+1.$$

Similarly $\quad H_i(W,M') = 0$ for $i \neq n, n+1$.

This shows the cobordism decomposes as a handlebody with only n- and (n+1)-handles, arrange these in ascending order:



For some intermediate N^{2n}.

We can view N as both k (n-1)-surgeries on M and ℓ (n-1)-surgeries on M'

$$\Rightarrow \quad M \# \left(\#_k S^n\times S^n\right) \cong M' \# \left(\#_\ell S^n\times S^n\right)$$

$$\Rightarrow \quad (K_n(M),\lambda,\mu) = (K_n(M'),\lambda',\mu') \in L_{2n}(\mathbb{Z}\pi_1(X))$$

□

Corollaries
- We can define (well) $[K_n,\lambda,\mu]$ for stably f.g. free K_n by stabilising via (n-1)-surgeries.
- We can define an element of the L-group ~~For non~~ without insisting n-connected as the surgeries to kill kernels below middle dim are deg 1 normal bordisms. Call this $\sigma_*(f,b)$

~~Theorem For~~ $\cancel{(f,b): M^{2n} \to X^{2n}}$ ~~a deg 1, normal map~~

$$\cancel{\sigma_*(f,b) = 0 \in L}$$

~~The map~~

Recall the normal structure set $\mathcal{J}(X)$ is the set of vector bundle reductions of SNF of X and that is in natural 1:1 correspondence with normal bordism classes of deg 1 normal maps $(f,b): M \to X$. We have shown

Theorem There is a well defined function

$$\sigma_*: \mathcal{J}(X^{2n}) \longrightarrow L_{2n}(\mathbb{Z}\pi_1(X))$$

such that $\sigma_*(x)$ vanishes iff x can be represented by (f,b) a htpy equivalence.

$\square$

We have also now proven

Theorem (Browder '82)

2n-dim GPC X for $n \geq 3$ with $\pi_1(X) = 0$ is htpc to a 2n-dim manifold iff

(i) $\mathcal{J}(X)$ non-empty i.e. $\exists\ (f,b): M^{2n} \longrightarrow X^{2n}$

and (ii)

(a) n even : $\text{signature}(H_n^{(M)}, \lambda) = \text{signature}(H_n(X), \lambda)$

(b) n odd : $\text{Arf}(K_{2k+1}(M; \mathbb{Z}_2), \lambda, \mu) = 0$

Proof (a) needs a little explanation

By connectivity (WLOG) we have an SES of $\mathbb{Z}\pi = \mathbb{Z}$-modules:

$$0 \to K_n(M) \to H_n(M) \to H_n(X) \to 0.$$

If we tensor with $\mathbb{R}$, this sequence splits as

$$H_n(M)\otimes\mathbb{R} \cong (H_n(X)\otimes\mathbb{R}) \oplus (K_n(M)\otimes\mathbb{R})$$

and so $\operatorname{sig}(M) = \operatorname{sig} X + \underbrace{\operatorname{sig} K_n(M)}_{\text{Vanishes} \Leftrightarrow \text{htpy equiv.}}$

<u>Example</u> Recall a framing for M^{2n} is $M \hookrightarrow S^{2n+k}$ and a framing $b: \nu_M \xrightarrow{\simeq} \varepsilon^k$ of the normal bundle.

This determines a deg 1 normal map to $X = S^m = M/cl(M \setminus D^{2n})$

i.e. collapse M to the boundary of a disk in M

Then $f: M \to S^m$ pulls back

$\varepsilon^\infty : S^m \to BO$ to

$b: f^*\varepsilon^\infty \xrightarrow{\simeq} \nu_M$

Hence we have morphisms

$$\sigma_*^{fr} : \Omega^{fr}_{2n} \to L_{2n}(\mathbb{Z})$$

For <u>n even</u> $\sigma_*(K_n(M)) = 0 \Leftrightarrow \sigma(M) = 0$ (as $\sigma(S^{2n}) = 0$)

For <u>n odd</u> define the Kervaire invariant of (M, b) to be

$$\sigma_*^{fr}(M, b) \in \mathbb{Z}_2$$

the Arf invariant of $(H_n(M;\mathbb{Z}_2); \lambda; \mu)$.

I.e.: σ_*^{fr} surjective → For $n = 1, 3, 7, 15, 31$ can write down Kervaire inv 1 (M, b)

For $n = 63$ <u>Still open</u>

O/w Hill, Hopkins, Ravenel (2009) showed $\nexists$ Kervaire invariant 1 (M, b).