Patrick Orson

The Kernel Formation

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Unles otherwise specified, $n \geq 2$.

⓪ Outline

Our data will be a deg. 1 n-conn normal map $(f,b): M^{2n+1} \longrightarrow X^{2n+1}$. We want to make it $(n+1)$-conn as then it will be a htpy equivalence. So we want to kill the non-vanishing groups $K_n(M)$ and $K_{n+1}(M)$.

Using a choice of "Heegard splitting" we will associate a $(-1)^n$-quad kernel formation $(K(M), \lambda, \mu; F, G)$ to (f,b) so that

$$K_n(M) = K/F+G \quad , \quad K_{n+1}(M) = F \cap G.$$

Recall that a formation is iso to trivial iff $K/F+G = F \cap G = 0$. Hence the question of whether (f,b) is a htpy equiv will be transferred to whether $(K(M), \lambda, \mu; F, G)$ is trivial (stably).

We then need to worry about

1. Choice of Heegard splitting
2. Effect of deg 1 normal bordism.

① Heegard splittings

Consider the handle decomp of a closed 3-manifold given by

$$M = \underbrace{\underbrace{M^0 \cup 1\text{-handles}}_{= M^1} \cup 2\text{ handles}}_{= M^2} \cup 3\text{-handles}$$

If M conn then WLOG $M^0 = D^3 =$ 3-handles.

Suppose we have g 1-handles. Framing of a k-handle in an n-manifold is given by $\pi_{k-1}(O(n-k))$. So we have

$$\pi_0(O(n-1)) = \pi_0(O(2)) = \mathbb{Z}_2$$

So we have the orientable and non-orientable framings.

By PD there are g 2-handles and

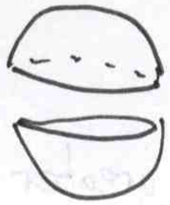
$$\pi_1(O(n-2)) = \pi_1(O(1)) = 0$$

$\Rightarrow$ no choices in framing.

So for orientable manifolds the only choices are

- # 1-handles
- attaching circles of 2-handles $\begin{pmatrix}\text{1 pair for each} \\ \text{handle}\end{pmatrix}$

<u>e.g.</u> <u>$g = 0$</u>



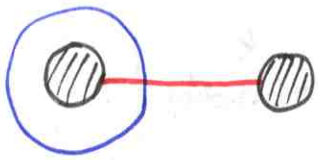
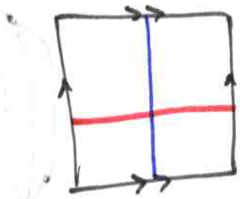
$= S^3$ (in fact this is only $g=0$)

<u>$g = 1$</u>

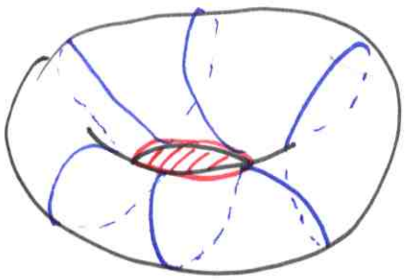
$= S^3$ This is how you draw the decomp $S^3 = \partial(D^2 \times D^2)$

$= S^1 \times D^2 \cup_{T^2} D^2 \times S^1$

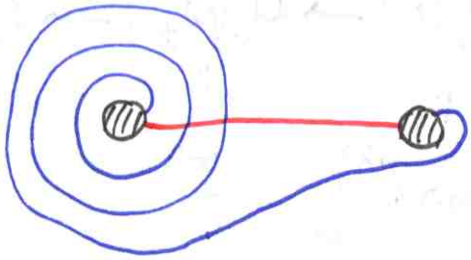
or  draw as 

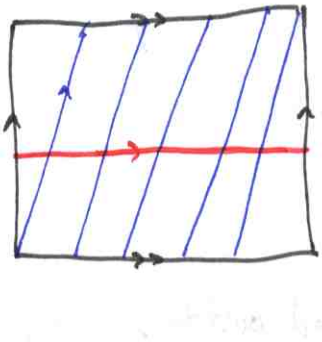
then $S^3 =$  or 

<u>$g = 1$</u>



Lens spaces.

$=$ 

$=$ 

<u>Remark</u> • $g \geq \mathrm{rk}(H_1(M))$

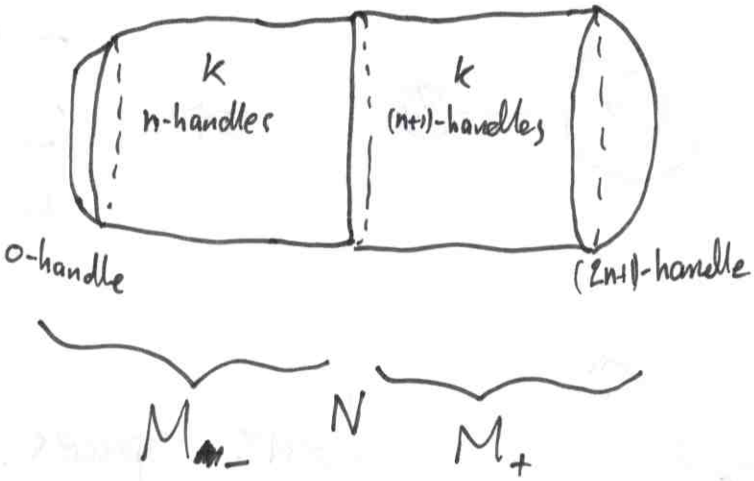
• $\exists M$ s.t. minimal g is greater than $H_1(M)$. (Boileau/Zieschang)

• These diagrams allow you to compute combinatorial HF-homology.

Now let M be closed, $(n-1)$-conn, $(2n+1)$-dim. We have a handle decomp:

$$N = \partial M_\pm = \#_k S^n \times S^n$$

$$M_\pm \simeq \#_k S^n \times D^{n+1}$$



Consider LES splits:

$$0 \to \underset{\substack{\| \\ 0}}{H_{n+1}(M_\pm)} \to \underset{\substack{\| \\ \mathbb{Z}^g}}{H_{n+1}(M_\pm, N)} \xrightarrow{\partial} \underset{\substack{\| \\ H_{(-1)^n}(\mathbb{Z}^k)}}{H_n(N)} \xrightarrow{i} \underset{\substack{\| \\ \mathbb{Z}^g}}{H_n(M_\pm)} \to 0$$

So $\ker i = \operatorname{im} \partial = L_\pm$ are the Lagrangians of $H_{(-1)^n}(\mathbb{Z}^k)$.

<u>Example</u> For our lens spaces before we have

$$H_n(N) = H_1(T^2) = \langle \alpha, \beta \rangle$$

$M_- = S^1 \times D^2$, $M_+ = D^2 \times S^1$ with generators glued outside as α, β.

$\mathrm{im}(\partial_-) = \alpha$ as this was the non-collapsing generator

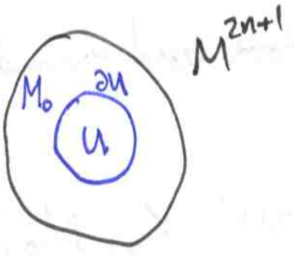
$\mathrm{im}(\partial_+) = \alpha + q\beta$ as the core of the outside torus wraps q times around.

$\rightarrow \left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} ; \langle \alpha \rangle , \langle \alpha \rangle \oplus q \langle \beta \rangle \right)$ is the symmetric formation.

<u>Remark</u> Another way to draw the Heegaard splitting is

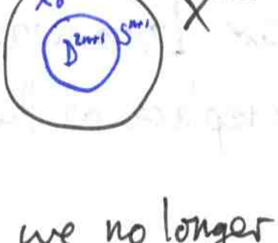
We wish to include normal data:

$$\begin{array}{c} (M_0, \partial U) \\ (f_0, b_0) \downarrow \\ (X_0, S^{2n+1}) \end{array}, \qquad \begin{array}{c} (U, \partial U) \\ (e, a) \downarrow \\ (D^{2n+1}, S^{n+1}) \end{array}$$



$$\boxed{\begin{array}{c} U = \#_k S^n \times D^{n+1} \\ \partial U = \#_k S^n \times S^n \end{array}}$$

$\downarrow (f, b)$ deg 1 normal



Unlike the even-dim case we no longer need to worry about whether generators $x_1, \dots, x_k$ of $K_n(M)$ can be killed by surgery as we are in the range to ensure framed embeddings:

$$\begin{tikzcd} S^n \times D^{n+1} \arrow[r, hook, "g_i"] \arrow[d] & M^{2n+1} \arrow[d, "f"] \\ D^{n+1} \times D^{n+1} \arrow[r, "h"] & X \end{tikzcd}$$

WLOG, assume all embeddings happen in small n'hoods in M and X. See the blue drawing above.

<u>Remark</u>
- If X is a homology sphere then $K_n(M) \cong H_n(M)$ and this is a Heegard splitting of M. (note this is happening for $(U, \partial U)$)
- If not, we will not have nice symmetric $M_\pm$ as before. But we will still have it in the kernel modules.

<u>Def'n</u> The kernel formation of (f, b) with respect to our choice of generators and embeddings is

$$\boxed{\left(K_n(\partial U), \lambda, \mu;\ \underset{\underset{F}{\shortparallel}}{K_{n+1}(U, \partial U)},\ \underset{\underset{G}{\shortparallel}}{K_{n+1}(M_0, \partial U)}\right)}$$

The argument that F and G are Lagrangians is the same as without normal data except we replace all the homology groups with kernel modules.

As promised, this has the required property:

<u>Proposition</u> As $\mathbb{Z}\pi$-modules:

$$K_n(M) \cong K/_{F+G}\ , \quad K_{n+1}(M) \cong F \cap G.$$

<u>Proof</u> We have LES:

$$\begin{tikzcd}
0 \arrow[r] & K_{n+1}(M) \arrow[r] & K_{n+1}(M,U) \arrow[r] \arrow[d, dashed, "\cong\ \text{excision}"] & K_n(U) \arrow[r] \arrow[d, dashed, "\cong"] & K_n(M) \arrow[r] & 0 \\
& & K_{n+1}(M_0,\partial U) \arrow[r] \arrow[d, equal] & K_n(\partial U)/K_{n+1}(U,\partial U) \arrow[d, equal] & & \\
& & G & K/_F & &
\end{tikzcd}$$

↖ by splitting of LES of $(U,\partial U)$.

□

Now we are in a position to examine the effect on the formation of:

1. Choice of Heegard splitting
2. Deg 1 normal bordism to another n-connected M'^{2n+1}.

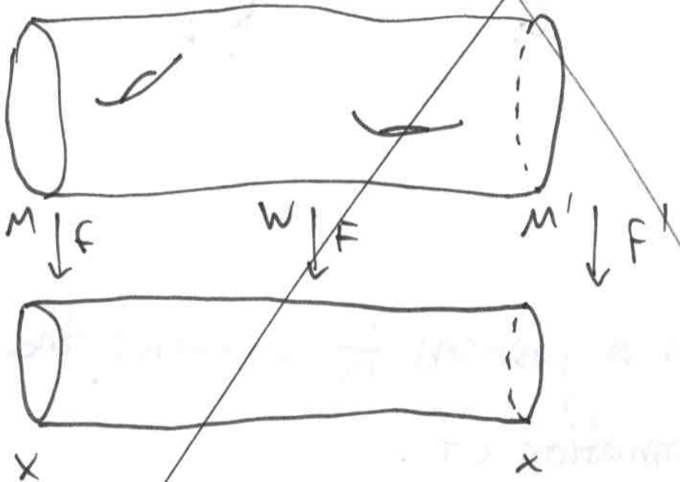
It turns out they are essentially the same thing.

Proposition A deg 1 normal, n-conn map $(f,b) : M^{2n+1} \to X^{2n+1}$ uniquely determines a stable iso class of formation.

i.e. $\left\{ \begin{array}{l} \text{choice of} \\ \text{splitting} \end{array} \right\} \xleftrightarrow{1:1} \left\{ \begin{array}{l} \text{choice of rep} \\ \text{of stable} \\ \text{class} \end{array} \right\}$

Proof (later) $\square$

A Heegaard splitting given by $x_1, \dots, x_k \in K_n(M)$, the generators, determines k simultaneous surgeries with trace W and effect M'. WLOG, (f',b') is n-conn and (F,B) is $(n+1)$-conn.



So a choice of splitting gives such a cobordism.
The converse is also true.

Claim $\left\{ \begin{array}{l} (2n+2)\text{-dim} \\ (n+1)\text{-conn, deg } 1 \\ \text{normal bordism from} \\ (M, f, b) \end{array} \right\} \xleftrightarrow{1:1} \left\{ \begin{array}{l} \text{choice of rep} \\ \text{of stable} \\ \text{class of formation} \end{array} \right\}$

Proposition A degree 1, n-conn, normal map $(f,b): M^{2n+1} \longrightarrow X^{2n+1}$ uniquely determines a stable iso class of $(-1)^n$-quadratic formations. Moreover:

$$\left\{\begin{matrix}\text{Heegard splittings}\\ \text{of } (f,b)\end{matrix}\right\} \xleftrightarrow{1:1} \left\{\begin{matrix}\text{reps of the stable}\\ \text{class}\end{matrix}\right\}$$

Proof We need to prove the two choices we made do not affect the stable class. That is:

(a) Choice of embeddings $g_i: S^n \times D^{n+1} \hookrightarrow M$

(b) Choice of generators $x_i \in K_n(M)$.

(a) Different embeddings give isomorphic formations:

First, $K_n(M) = \pi_{n+1}(f) = I_{n+1}(f)$, the last equality coming from the LES

$$\dots \to \pi_n(V_{m-n}) \to I_{n+1}(f) \to \pi_{n+1}(f) \to \pi_{n-1}(V_{m-n}) \to \dots$$

and setting $m = 2n+1$; $\pi_n(V_{n+1}) = \pi_{n-1}(V_{n+1}) = 0$. So fix an Heegard splitting $x_1, \dots, x_k$ with $g_1, g_2, \dots, g_k$ representatives, there exist regular homotopies $d_1, \dots, d_k$ to a different choice of embeddings $g_1', g_2', \dots, g_k'$. ~~that gw~~ We have two splittings

$$(f,b) = \begin{cases} (f_0, b_0) \cup (e, a) : M_0 \cup U \longrightarrow X_0 \cup D^{2n+1} \\ (f_0', b_0') \cup (e', a') : M_0' \cup U' \longrightarrow X_0' \cup D^{2n+1} \end{cases}$$

With formations

$$(K_n(\partial U), \lambda, \mu \; ; \; K_{n+1}(U, \partial U), K_{n+1}(M_0, \partial U))$$

$$(K_n(\partial U'), \lambda', \mu' ; \; K_{n+1}(U', \partial U'), K_{n+1}(M_0', \partial U'))$$

and maps

$$\begin{pmatrix} \gamma \\ \delta \end{pmatrix} : G \longrightarrow F \oplus F^*$$

$$\begin{pmatrix} \gamma' \\ \delta' \end{pmatrix} : G' \longrightarrow F' \oplus (F^*)'$$

As the two cases are the same to regular htpy we have isos:

$$\alpha : F \longrightarrow F' \qquad \text{and a } \underline{\text{track}} \qquad \coprod_{i=1}^{k} S^n \times D^{n+1} \times I \looparrowright M \times I$$

$$\beta : G \longrightarrow G' \qquad \text{for } d_i$$

with $(-1)^{n+1}$-quad form

and hence a commutative square

$$(F^*, \psi)$$

$$\shortparallel$$

$$K_{n+1}(U, \partial U)^*$$

$$\begin{array}{ccc} G & \xrightarrow{\quad\beta\quad} & G' \\ {\scriptstyle\begin{pmatrix} \gamma \\ \delta \end{pmatrix}}\downarrow & & \downarrow{\scriptstyle\begin{pmatrix} \gamma' \\ \delta' \end{pmatrix}} \\ F \oplus F^* & \longrightarrow & F' \oplus (F')^* \end{array}$$

where the lower map is

$$\begin{pmatrix} \alpha & \alpha(\psi + (-1)^{n+1}\psi^*) \\ 0 & (\alpha^*)^{-1} \end{pmatrix}$$

Which defines an iso $\quad (K_n(\partial U), \lambda, \mu) \longrightarrow (K_n(\partial U'), \lambda', \mu')$

$$\text{s.t.} \quad F \longrightarrow F'$$

$$G \longrightarrow G'$$

(b) A different choice of generators gives a stably iso formation:

Let the two choices by

(i) $\{x_1, \dots, x_k\}$ where $l \geq x$.

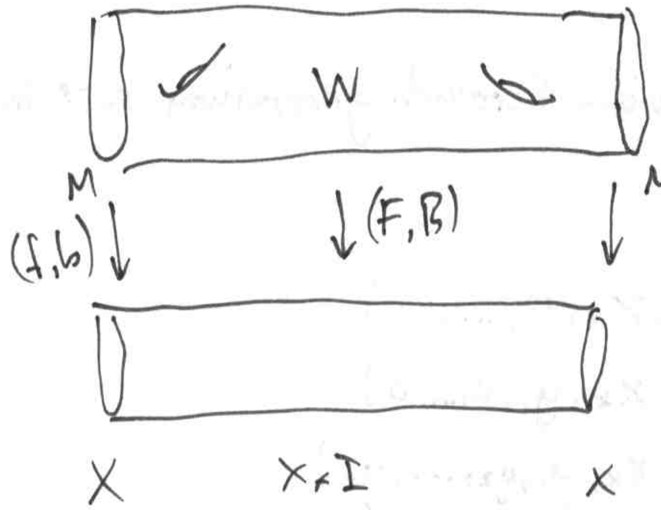
(ii) $\{y_1, \dots, y_l\}$

There is an algorithm for moving through generating sets to get from (i) to (ii):

$$
\begin{aligned}
(x_1, \dots, x_k) &\longmapsto (x_1, \dots, x_k, 0, \dots, 0) \\
&\longmapsto (x_1, \dots, x_k, y_1, 0, \dots, 0) \\
&\longmapsto (x_1, \dots, x_k, y_1, y_2, \dots, y_l) \\
&\longmapsto (y_1, \dots, y_l, x_1, \dots, x_k) \\
&\longmapsto (y_1, \dots, y_l, x_1, \dots, x_{k-1}, 0) \\
&\longmapsto (y_1, \dots, y_l, 0, 0, \dots, 0) \\
&\longmapsto (y_1, \dots, y_l)
\end{aligned}
$$

Procedure	Effect on formation	Geometric operation
Adding / removing zeroes.	Adding / removing $(H_{(-1)^n}(\mathbb{Z}\pi, X); \mathbb{Z}\pi, \mathbb{Z}\pi^*)$	Using / not using a triv embedded S^n. i.e. this is 0-surgery in K_n.
Adding lin combos of x_i.	None	Doing surgery on the corresponding conn sum (addition in $I_{n+1}(f)$)
Permuting elements	None	None

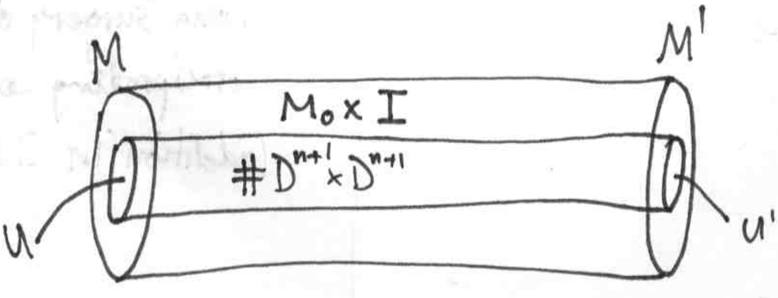
A Heegard splitting with generators $x_1, \dots, x_k \in K_n(M)$ determines k simultaneous surgeries with trace W and effect M'. We may surger inside W^{2n+2} to make (F,B) $(n+1)$-conn:



Such a W is called a *presentation* of (f,b). In terms of this, we may write the formation of the Heegard splitting as

$$\left(H_{(-1)^n}(K_{n+1}(W,M')) \,;\, K_{n+1}(W,M'),\, K_{n+1}(W)\right).$$

and this is an isomorphic formation by excision and recalling that all the surgery was happening in U so we have a lot of symmetry:

$W =$ 

In fact we have a presentation of (f',b') with formation

$(\Leftarrow)$ Suppose a stable iso $(K_n(\partial U), \lambda, \mu ; F, G) \oplus T^{r} \cong \partial(K_W, \lambda_W, \mu_W) \oplus T^{r}$

Realise (K_W, λ_W, μ_W) by a deg 1 normal bordism W^{2n+2} that is $(n+1)$-conn a presentation for (f,b). In this presentation (f',b') a htpy equiv.

$\square$

$$\left(H_{(-1)^n}(K_{n+1}(W,M));\ K_{n+1}(W,M),\ K_{n+1}(W)\right) = \left(H_{(-1)}(F);\ F^*, G\right)$$

$$\|$$

$$\left(H_{(-1)^n}(K_{n+1}(U',\partial U'));\ K_{n+1}(U',\partial U'),\ K_{n+1}(\overset{M_0}{\overset{\|}{M_0'}}, \overset{\partial U}{\overset{\|}{\partial U'}})\right)$$

We have inclusions $\begin{pmatrix}\gamma\\ \delta\end{pmatrix} : G \longrightarrow F \oplus F^*$

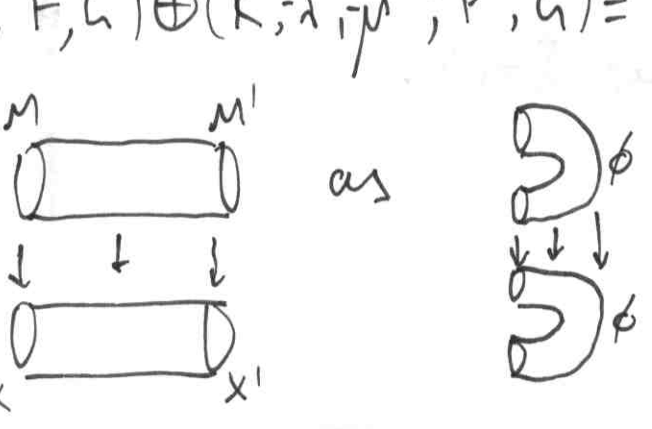
and $\begin{pmatrix}\gamma'\\ \delta'\end{pmatrix} : \overset{G}{\overset{\|}{G'}} \longrightarrow \underset{F'}{\underset{\|}{F^* \oplus F}}$ by $\begin{pmatrix}\gamma'\\ \delta'\end{pmatrix} = \underbrace{\begin{pmatrix}(-1)^{n+1}\delta\\ \gamma\end{pmatrix}}_{\text{in terms of the maps for } (f,b).}$

What of bordism of formations?

<u>Claim</u> (f,b) an n-conn, deg 1 normal map is deg 1 bordant to a htpy equivalence (f',b') iff for some Heegard splitting we have $(K_n(\partial U), \lambda, \mu; F, G)$ stably iso to $\partial(K, \lambda_w^*, \mu_w')$ some $(-1)^{n+1}$-quadratic form.

<u>Cor</u> For a presentation $(W, \bar{M}' \amalg M)$ the kernel formations of the M and M' are related by a stable iso

$$(K, \lambda, \mu;\ F, G) \oplus (K', -\lambda', -\mu';\ F', G') \cong \partial(K_w, \lambda_w, \mu_w)$$

<u>Proof</u> Regard  □

Proof of Claim ($\Longrightarrow$) Suppose $(K', \lambda', \mu'; F', G')$ is stab triv.

$$= (H_{(-1)^n}(F); F^*, G)$$

$$\Rightarrow G \cong F$$

$$\Rightarrow \gamma \text{ an iso and } \delta = 0.$$

$\gamma : K_{n+1}(W) \xrightarrow{\cong} K_{n+1}(W, M')$ so STP that $\gamma(\Gamma(K_{n+1}(W))) = K_{n+1}(W)$

as $\partial(K_W, \lambda_W, \mu_W) = (H_{(-1)^n}(K_{n+1}(W)); K_{n+1}(W), \Gamma(K_{n+1}(W)))$

But recall the definition of

$$\Gamma(K_{n+1}(W)) := \{(x, \lambda_W(x))\} \subset K_{n+1}(W) \oplus K_{n+1}(W)^* = F \oplus F^*$$

But in fact $\lambda_W = \delta$ on $K_{n+1}(W)$ as δ defined by:

$$\delta : K_{n+1}(W) \xrightarrow{i} K_{n+1}(W, M) \xrightarrow{PD} K^{n+1}(W, M')$$

$$\xrightarrow{\cong} K_{n+1}(W, M')^* \quad \text{(canonical iso)}$$

~~$K_{n+1}(W, M)^*$ (by symmetry in M, M')~~

$$\xrightarrow{\cong} K_{n+1}(W)^*$$

where the last iso is from the fact that $\gamma : K_{n+1}(W) \xrightarrow{\cong} K_{n+1}(W, M')$ is an iso and so we get $K_{n+1}(W)^* = K_{n+1}(W, M')^*$.

So $\lambda_W = 0$ as $\delta = 0$.