A brief introduction to surgery theory

A 1-lecture reduction of a 3-lecture course given at Cambridge in 2005

http://www.maths.ed.ac.uk/~aar/slides/camb.pdf

Andrew Ranicki

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Time scale

1905 m-manifolds, duality (Poincaré)

1910 Topological invariance of the dimension m of a manifold (Brouwer)

1925 Morse theory

1940 Embeddings (Whitney)

1950 Structure theory of differentiable manifolds, transversality, cobordism (Thom)

1956 Exotic spheres (Milnor)

1962 h-cobordism theorem for $m \geqslant 5$ (Smale)

1960's Development of surgery theory for differentiable manifolds with $m \geqslant 5$ (Browder, Novikov, Sullivan and Wall)

1965 Topological invariance of the rational Pontrjagin classes (Novikov)

1970 Structure and surgery theory of topological manifolds for $m \geqslant 5$ (Kirby and Siebenmann)

1970– Much progress, but the foundations in place!

The fundamental questions of surgery theory

- Surgery theory considers the existence and uniqueness of manifolds in homotopy theory:
  1. **When is a space homotopy equivalent to a manifold?**
  2. **When is a homotopy equivalence of manifolds homotopic to a diffeomorphism?**
- Initially developed for differentiable manifolds, the theory also has PL(= piecewise linear) and topological versions.
- Surgery theory works best for $m \geqslant 5$: 1-1 correspondence

  geometric surgeries on manifolds

  $\sim$ algebraic surgeries on quadratic forms

  and the fundamental questions for topological manifolds have algebraic answers.
- Much harder for $m = 3, 4$: no such 1-1 correspondence in these dimensions in general.
- Much easier for $m = 0, 1, 2$: don't need quadratic forms to quantify geometric surgeries in these dimensions.

The unreasonable effectiveness of surgery

- *The unreasonable effectiveness of mathematics in the natural sciences* (title of 1960 paper by Eugene Wigner).
- Surgery is a drastic topological operation on manifolds, e.g. destroying connectivity.
- Given this violence, it is surprising that it can be used to distinguish manifold structures within a homotopy type, i.e. to answer the fundamental questions for $m \geqslant 5$!

The main ingredients of surgery theory

1. **Handlebody theory**: handles $D^i \times D^{m-i}$ attached at $S^{i-1} \times D^{m-i}$, are the building blocks of m-manifolds.
2. **Vector bundles**: the K-theory of vector bundles, such as the normal bundles $\nu_M : M \to BO(k)$ of embeddings of m-manifolds $M \subset S^{m+k}$,
3. **Quadratic forms**: the algebraic L-theory of quadratic forms, such as arise from the Poincaré duality of an m-manifold M

   $$H^{m-*}(M) \cong H_*(M)$$

   and the geometric interpretation using intersection numbers.
4. **The fundamental group**: need to consider Poincaré duality and quadratic forms over the ring $\mathbb{Z}[\pi_1(M)]$. In the non-simply-connected case $\pi_1(M) \neq \{1\}$ this could be quite complicated!

Surgery

- Given a differentiable m-manifold M^m and an embedding

$$S^i \times D^{m-i} \subset M \quad (-1 \leqslant i \leqslant m)$$

define the m-manifold obtained from M by **surgery**

$$M' \ = \ (M - S^i \times D^{m-i}) \cup D^{i+1} \times S^{m-i-1} \ .$$

- **Example** Let K, L be disjoint m-manifolds, and let $D^m \subset K$, $D^m \subset L$. The effect of surgery on $S^0 \times D^m \subset M = K \sqcup L$ is the **connected sum** m-manifold

$$K \# L \ = \ (K - D^m) \cup [0,1] \times S^{m-1} \cup (L - D^m) \ .$$

Surgery on surfaces

- Surface = 2 -manifold
- **Standard example** The effect of surgery on $S^0 \times D^2 \subset S^2$ is either a torus $S^1 \times S^1$ or a Klein bottle, according to the two orientations.
- **Proposition** Every orientable surface can be obtained from $\emptyset$ by a sequence of surgeries.
- **Proposition** A nonorientable surface M can be obtained from $\emptyset$ by a sequence of surgeries if and only if the Euler characteristic $\chi(M)$ is even.

Attaching handles

- Let L be an $(m+1)$-manifold with boundary ∂L. Given an embedding
$$S^i \times D^{m-i} \subset \partial L$$
define the $(m+1)$-manifold
$$L' = L \cup_{S^i \times D^{m-i}} h^{i+1}$$
obtained from L by **attaching an** $(i+1)$**-handle**
$$h^{i+1} = D^{i+1} \times D^{m-i} .$$
- **Proposition** The boundary $\partial L'$ is obtained from ∂L by surgery on $S^i \times D^{m-i} \subset \partial L$, and there is a homotopy equivalence
$$L' \simeq L \cup_{S^i} D^{i+1} .$$
The homotopy theoretic effect of attaching an $(i+1)$-handle is to attach an $(i+1)$-cell.

The trace

- The **trace** of the surgery on $S^i \times D^{m-i} \subset M^m$ is the elementary $(m+1)$-dimensional cobordism $(W; M, M')$ obtained from $M \times [0,1]$ by attaching an $(i+1)$-handle

$$W = (M \times [0,1]) \cup_{S^i \times D^{m-i} \times \{1\}} h^{i+1}$$

- **Proposition** An $(m+1)$-dimensional cobordism $(W; M, M')$ admits a Morse function $(W; M, M') \to ([0,1]; \{0\}, \{1\})$ with a single critical value of index $i+1$ if and only if $(W; M, M')$ is the trace of a surgery on an embedding $S^i \times D^{m-i} \subset M$.

Handle decomposition

- A **handle decomposition** of an $(m+1)$-dimensional cobordism $(W; M, M')$ is an expression as a union of elementary cobordisms

  $$(W; M, M') = (W_0; M, M_1) \cup (W_1; M_1, M_2) \cup \cdots \cup (W_k; M_k, M')$$

  such that

  $$W_r = (M_r \times [0,1]) \cup h^{i_r+1}$$

  is the trace of a surgery on $S^{i_r} \times D^{m-i_r} \subset M_r$ with

  $$-1 \leqslant i_0 \leqslant i_1 \leqslant \cdots \leqslant i_k \leqslant m .$$

- Note that M or M' (or both) could be empty.
- Handle decompositions non-unique, e.g. handle cancellation

  $$W \cup h^{i+1} \cup h^{i+2} = W$$

  if one-point intersection

  $$(\{0\} \times S^{m-i-1}) \cap (S^{i+1} \times \{0\}) = \{*\} \subset \partial(W \cup h^{i+1}) .$$

Cobordism = sequence of surgeries

- **Theorem** (Thom, Milnor 1961) Every $(m+1)$-dimensional cobordism $(W; M, M')$ admits a handle decomposition,

$$W \; = \; (M \times [0,1]) \cup \bigcup_{j=0}^{k} h^{i_j+1}$$

  with $-1 \leqslant i_0 \leqslant i_1 \leqslant \cdots \leqslant i_k \leqslant m$.

- **Proof** For any cobordism $(W; M, M')$ there exists a Morse function

$$f \; : \; (W; M, M') \to ([0,1]; \{0\}, \{1\})$$

  with critical values $c_0 < c_1 < \cdots < c_k$ in $(0,1)$: there is one $(i+1)$-handle for each critical point of index $i+1$.

- **Corollary** Manifolds M, M' are cobordant if and only if M' can be obtained from M by a sequence of surgeries.

Poincaré duality

- **Theorem** For any oriented $(m+1)$-dimensional cobordism $(W; M, M')$ cap product with the fundamental class $[W] \in H_{m+1}(W, M \cup -M')$ is a chain equivalence

$$[W] \cap - \ : \ C(W,M)^{m+1-*} \ = \ \mathrm{Hom}_{\mathbb{Z}}(C(W,M),\mathbb{Z})_{*-m-1} \xrightarrow{\simeq} C(W,M')$$

  inducing isomorphisms

$$H^{m+1-*}(W,M) \ \cong \ H_*(W,M') \ .$$

- **Proof** Compare the handle decompositions given by any Morse function

$$f \ : \ (W; M, M') \to ([0,1]; \{0\}, \{1\})$$

  and the dual Morse function

$$1-f \ : \ (W; M', M) \to ([0,1]; \{0\}, \{1\}) \ .$$

- For $M = M' = \emptyset$ have $H^{m+1-*}(W) \cong H_*(W)$.

The algebraic effect of a surgery

- **Proposition** If $(W; M, M')$ is the trace of a surgery on $S^i \times D^{m-i} \subset M^m$ there are homotopy equivalences

$$M \cup D^{i+1} \simeq W \simeq M' \cup D^{m-i} .$$

Thus M' is obtained from M by first attaching an $(i+1)$-cell and then detaching an $(m-i)$-cell, to restore Poincaré duality.

- **Corollary** The cellular chain complex $C(M')$ is such that

$$C(M')_r = \begin{cases} C(M)_r \oplus \mathbb{Z} & \text{for } r = i+1, m-i-1 \text{ distinct} , \\ C(M)_r \oplus \mathbb{Z} \oplus \mathbb{Z} & \text{for } r = i+1 = m-i-1 , \\ C(M)_r & \text{otherwise} \end{cases}$$

with differentials determined by the i-cycle $[S^i] \in C(M)_i$ and the Poincaré dual $(m-i)$-cocycle

$$[S^i]^* \in C(M)^{m-i} = \mathrm{Hom}_{\mathbb{Z}}(C(M)_{m-i}, \mathbb{Z}) .$$

Poincaré complexes: definition

- An m-**dimensional Poincaré complex** X is a finite CW complex with a homology class $[X] \in H_m(X)$ such that there are Poincaré duality isomorphisms

$$[X] \cap - \; : \; H^{m-*}(X) \; \cong \; H_*(X)$$

  with arbitrary coefficients.
- Similarly for an m-**dimensional Poincaré pair** $(X, \partial X)$, with $[X] \in H_m(X, \partial X)$ and

$$[X] \cap - : H^{m-*}(X) \; \cong \; H_*(X, \partial X) \; .$$

- If X is simply-connected, i.e. $\pi_1(X) = \{1\}$, it is enough to just use $\mathbb{Z}$-coefficients.
- For non-oriented X need twisted coefficients.

Poincaré complexes: examples

- An m-manifold is an m-dimensional Poincaré complex.
- A finite CW complex homotopy equivalent to an m-dimensional Poincaré complex is an m-dimensional Poincaré complex.
- If M_1, M_2 are m-manifolds with boundary and $h : \partial M_1 \simeq \partial M_2$ is a homotopy equivalence then $X = M_1 \cup_h M_2$ is an m-dimensional Poincaré complex. If h is homotopic to a diffeomorphism then X is homotopy equivalent to an m-manifold.
- Conversely, if X is not homotopy equivalent to an m-manifold then h is not homotopic to a diffeomorphism.

Poincaré complexes vs. manifolds

- **Theorem** Let $m = 0, 1$ or 2.
  (i) Every m-dimensional Poincaré complex X is homotopy equivalent to an m-manifold. (Non-trivial for $m = 2$).
  (ii) Every homotopy equivalence $M \to M'$ of m-manifolds is homotopic to a diffeomorphism.
- Theorem is false for $m \geqslant 3$.
- (Reidemeister, 1930) Homotopy equivalences $L \simeq L'$ of 3-dimensional lens spaces $L = S^3/\mathbb{Z}_p$ which are not homotopic to diffeomorphisms. (Lens spaces classified by Whitehead torsion).

Homotopy types of manifolds

- The **manifold structure set** $\mathcal{S}(X)$ of an m-dimensional Poincaré complex X is the set of equivalence classes of pairs (M, h) with M an m-manifold and $h : M \to X$ a homotopy equivalence, subject to

  $$(M, h) \sim (M', h')$$

  if $h^{-1}h' : M' \to M$ is homotopic to a diffeomorphism.
- **Existence Problem** Is $\mathcal{S}(X)$ non-empty?
- **Uniqueness Problem** If $\mathcal{S}(X)$ is non-empty, compute it by algebraic topology.
- There are two versions: $\mathcal{S}^O(X)$ for differentiable manifolds and $\mathcal{S}^{TOP}(X)$ for topological manifolds.
- $\mathcal{S}^0(S^m) =$ exotic differentiable structures on S^m (Milnor 1956, Kervaire-M 1963)
- For $m \geqslant 5$ $\mathcal{S}^{TOP}(S^m) = 0$ (Generalized Poincaré conjecture, Smale 1962)

The h-cobordism theorem

- **Theorem** (Smale, 1962) Let $(W; M, M')$ be an $(m+1)$-dimensional h-cobordism, so that the inclusions $i : M \subset W$, $i' : M' \subset W$ are homotopy equivalences. If $m \geqslant 5$ and W is simply-connected then $(W; M, M')$ is diffeomorphic to $M \times ([0,1]; \{0\}, \{1\})$ with the identity on M. In particular, the homotopy equivalence $h = i^{-1}i' : M' \to M$ is homotopic to diffeomorphism, and

$$(M', h) \;=\; (M, 1) \in \mathcal{S}(M) \;.$$

- Need $m \geqslant 5$ for 'Whitney trick' realizing algebraic moves by handle cancellations.
- The non-simply-connected version is called the s-**cobordism theorem** (Barden, Mazur and Stallings, 1964), and requires the Whitehead torsion condition

$$\tau(i) \;=\; \tau(i') \;=\; 0 \in Wh(\pi_1(M)) \;.$$

The Hirzebruch signature theorem

- The **signature** of an oriented $4k$-manifold M is the signature $\sigma(M) \in \mathbb{Z}$ of the intersection symmetric form $(H^{2k}(M), \lambda)$.
- **Theorem** (Hirzebruch, 1954) The signature is a characteristic number of the tangent bundle τ_M

$$\sigma(M) \ = \ \langle \mathcal{L}_k(p_1(M), p_2(M), \dots, p_k(M)), [M] \rangle \in \mathbb{Z}$$

with $\mathcal{L}_k$ a polynomial with rational coefficients in the Pontrjagin classes

$$p_i(M) \ = \ (-)^i c_{2i}(\tau_M \otimes \mathbb{C}) \in H^{4i}(M) \ .$$

- **Example** $\mathcal{L}_1 = p_1(M)/3$,
$\mathcal{L}_2 = (7p_2 - (p_1)^2))/45, \ \dots$ (Bernoulli numbers)

The converse of the signature theorem

- **Theorem** (Browder, 1962) For $k \geqslant 2$ a simply-connected $4k$-dimensional Poincaré complex X is homotopy equivalent to a manifold if and only if there exists a vector bundle $\eta \in \text{Vect}_j(X)$ with a map $\rho : S^{j+4k} \to T(\eta)$ with Hurewicz image
$$[\rho] \;=\; [X] \in \widetilde{H}_{j+4k}(T(\eta)) \;=\; H_{4k}(X)$$
such that
$$\sigma(X) \;=\; \langle \mathcal{L}_k(p_1(-\eta), \ldots, p_k(-\eta)), [X] \rangle \in \mathbb{Z}$$
with $\sigma(X)$ the signature of the intersection form $(H^{2k}(X), \lambda)$ and $-\eta$ any vector bundle over X such that $\eta \oplus -\eta$ is trivial.

The Browder-Novikov-Sullivan-Wall 2-stage obstruction

- The classical 1970 answer to the fundamental questions is described in the books of Browder and Wall, with a 2-stage obstruction:

1. a primary topological K-theory obstruction for the normal bundle
2. a secondary algebraic L-theory obstruction for the Poincaré duality.

- There is such a 2-stage obstruction for both differentiable and topological manifolds.
- Surprisingly, there is a simplification for topological manifolds, uniting the 2 stages in a single obstruction.
- *Topological manifolds bear the simplest possible relation to their underlying homotopy types* (Siebenmann, 1970)

The total surgery obstruction

- (R., 1980 –) Development of a single obstruction, uniting the two stages, using a covariant functor

$$\mathcal{S}_* \;:\; \{\text{topological spaces}\} \to \{\mathbb{Z}\text{-graded abelian groups}\}\ ;$$
$$X \to \mathcal{S}_*(X)\ .$$

- **Existence of a manifold structure** A finite CW complex X with m-dimensional Poincaré duality has a total surgery obstruction $s(X) \in \mathcal{S}_m(X)$. For $m \geqslant 5$ X is homotopy equivalent to a topological m-manifold if and only if $s(X) = 0 \in \mathcal{S}_m(X)$.
- **Uniqueness of manifold structures** A homotopy equivalence $f : M \to X$ of topological m-manifolds has a total surgery obstruction $s(X) \in \mathcal{S}_{m+1}(X)$. For $m \geqslant 5$ f is homotopic to a homeomorphism if and only if

$$s(f) \;=\; 0 \in \mathcal{S}_{m+1}(X) \;=\; \mathcal{S}^{TOP}(X)\ .$$

- *Algebraic L-theory and topological manifolds* (CUP, 1992)