Lecture 1 - Surgery, Handles and Morse Theory: Exercises

Surgery

1. Do there exist surgeries with effect diffeomorphic to the original manifold?
2. Prove that any finitely presented group is the fundamental group of some 4-manifold.
3. (a) Describe the effects on H_* and π_1 of surgery on 0, 1 and 2 dimensional manifolds.
   (b) Classify 2-manifolds by surgery hence proving the h-cobordism theorem for surfaces.
4. (a) Give examples of manifolds M^n and $\mathbb{Z}\pi_1$-modules $\langle x\rangle \subset \pi_k(M)$ for some k such that
      i. $\langle x\rangle$ is a free $\mathbb{Z}\pi_1$-module.
      ii. $\langle x\rangle$ is a stably free $\mathbb{Z}\pi_1$-module.
      iii. $\langle x\rangle$ is a projective $\mathbb{Z}\pi_1$-module.
   (b) How does being free/stably free/projective affect whether $\langle x\rangle$ can be killed by surgery?
5. (a) Let $\langle x\rangle \subset \pi_n(M^{2n})$ be killable by surgery. Describe the effects on H_{n-1}, H_n and H_{n+1} of surgery on x
   (b) Give necessary and sufficient conditions not to change H_{n-1} and H_{n+1}
6. **We should have a question about framings, though perhaps this would fit better in lecture 3 with the questions on knot surgeries?**

Morse Theory

7. (a) *(Part III - Morse Homology)* Show that being Morse is a C^2-open condition.
   (b) *(Part III - Morse Homology)* Prove that Morse functions are dense inside the space of continuous functions.
   (c) Are all Morse functions height functions for an appropriate embedding (into high dimensional Euclidean space for example)?
8. (a) Interpret cancelling handles and thus handle trading in terms of Morse functions.
   (b) Give an example of a Morse function on D^3 with a two 0-handles, two 1-handles and a single 2-handle such that either of the 1-handle cancels with any of the 0 or 2 handles.
9. (a) Explain, given the relationship between Morse functions and handle decompositions, how to determine the incidence numbers $\langle h^{k+1}_\alpha | h^k_\beta\rangle$ from the Morse function.
   (b) Give an explicit Morse function on $\mathbb{R}P^m$. Find with proof, the critical points and indices and then compute $H_*(\mathbb{R}P^m;\mathbb{Z})$.
10. *(Part III - Morse Theory)* Let $SO(n) = \{X \in M_n(\mathbb{R}) | X^tX = I, \det X = 1\}$ be the special orthogonal group. You may assume that $SO(n) \subset M_n(\mathbb{R}) = \mathbb{R}^{n^2}$ is a smooth submanifold and that the tangent space at X, $T_X SO(n)$, is the affine space $\{X + AX | A^t = -A\}$.

    Suppose $1 < c_1 < \ldots < c_n$, and let C be the diagonal matrix $\mathrm{diag}\{c_1, \ldots, c_n\}$. Define a function

$$\begin{aligned} f : SO(n) &\rightarrow \mathbb{R} \\ X &\mapsto \mathrm{tr}(CX) \end{aligned}$$

    Find the critical points of f and their indices. Deduce that $\chi(SO(3)) = 0$.