Lecture 2 - Orientations, Poincaré Duality, local coefficients and the handle chain complex: Exercises

Orientations

1. Let M be a CW complex. Recall we define the orientation character $\omega_1 : \pi_1(M) \to \mathbb{Z}_2$ as follows:

   Let $\gamma : S^1 \to M$ represent a class in $\pi_1(M)$, then consider the lift to the orientation double cover $M_{\mathbb{Z}_2}$:

$$\begin{array}{ccc} & & M_{\mathbb{Z}_2} \\ & \overset{\overline{\gamma}}{\nearrow} & \downarrow \\ S^1 & \xrightarrow{\gamma} & M \end{array}$$

   we define

$$\omega(\gamma) = \begin{cases} +1, & \overline{\gamma} \text{ exists} \\ -1, & \text{otherwise} \end{cases}$$

   (a) Prove that ω_1 is trivial if and only if $TM^{(1)}$ is trivial.

   (b) Formulate a definition of ω_k given $\omega_1, \ldots, \omega_{k-1}$ are all trivial such that ω_k is trivial if and only if any trivialisation of $TM^{(k-1)}$ can be extended to a trivialisation of $TM^{(k)}$.

2. **Perhaps a question about $\mathbb{C}$-orientations or $\mathbb{H}$-orientations?**

Poincaré Duality

3. For a simplicial complex X, define the *front p-face* of an n-simplex $\sigma = [v_0 \ldots v_n]$ as ${}_p\sigma := [v_0 \ldots v_p]$ and the *back q-face* as $\sigma_q := [v_{n-q} \ldots v_n]$.

   The *Alexander-Whitney diagonal approximation* is given by

$$\begin{array}{rcl} \tau : C_n(X) & \to & (C(X) \otimes C(X))_n \\ \sigma & \mapsto & \sum_{p+q=n} {}_p\sigma \otimes \sigma_q \end{array}$$

   and the *partial evaluation map* is defined as

$$\begin{array}{rcl} E : C^r(X) \otimes C_p(X) \otimes C_q(X) & \to & C_q(X) \\ a \otimes z \otimes w & \mapsto & \begin{cases} a(w) \otimes z, & r = q \\ 0, & \text{otherwise} \end{cases} \end{array}.$$

   Recall, we define the *cap product* on the chain level by

$$a \cap z := E(a \otimes \tau(z))$$

   and this descends to a well defined product on (co)homology.

   (a) Consider S^1 as a simplicial complex with three 0-simplices and three 1-simplices. Compute explicitly, using the Alexander-Whitney diagonal approximation, the map

$$- \cap [S^1] : C^{1-*}(S^1) \to C_*(S^1)$$

   thus verifying that S^1 has Poincaré duality.

   (b) **A 2nd example?**

4. Let M be

   (a) S^1

   (b) $S^1 \times S^1$

   (c) $\mathbb{R}P^2$

   (d) K the Klein bottle

   Consider all possible representations $\omega : \pi_1(M) \to \mathbb{Z}_2$. Compute

   - $H^*_{\mathbb{Z},\omega}(\widetilde{M}) := H^*(M; \mathbb{Z}\pi_1(M)_\omega)) = H_*(\mathrm{Hom}_{\mathbb{Z}\pi_1(M)}(C(\widetilde{M}), \mathbb{Z}\pi_1(M)_\omega))$.
   - $H_*^{\mathbb{Z},\omega}(\widetilde{M}) := H_*(M; \mathbb{Z}\pi_1(M)_\omega)) = H_*(C(\widetilde{M}) \otimes_{\mathbb{Z}\pi_1(M)} \mathbb{Z}\pi_1(M)_\omega)$.

   For what ω do we get Poincaré Duality

$$[M] \cap - : \begin{cases} H^k_{\mathbb{Z},\omega}(\widetilde{M}) \to H_{\dim M - k}(\widetilde{M}) \\ H^k(\widetilde{M}) \to H^{\mathbb{Z},\omega}_{\dim M - k}(\widetilde{M}) \end{cases} ?$$

   For S^1, why is the correct involution for Poincaré Duality

$$\Sigma a_j t^j \mapsto \Sigma a_j \omega(t) t^{-j}$$

   and not

$$\Sigma a_j t^j \mapsto \Sigma a_j \omega(t) t^j \ ?$$

Local coefficients

5. Prove that the two different points of view for local coefficients are equivalent.

6. Let A be a $\mathbb{Z}\pi_1(M)$-module. Construct a cover $\widetilde{M}$ such that the (co)homology of M with local coefficients in A is equal to the untwisted (co)homology of $\tilde{M}$.

The handle chain complex

7. Let M_1 be obtained from M_0 by b_0 k-surgeries with trace cobordism $(W_0; M_0, M_1)$. Let M_2 be obtained from M_1 by b_1 $(k+1)$-surgeries with trace cobordism $(W_1; M_1, M_2)$. Let $(W; M_0, M_2)$ be $W_1 \cup W_2$.

   Consider the relative handle chain complex

$$\begin{array}{ccc} C(W; M_0)_{k+1} & \xrightarrow{\partial_{k+1}} & C(W; M_0)_k \\ \| & & \| \\ H_{k+1}(W_1; M_1) = \mathbb{Z}^{b_1} & & H_k(W_0; M_0) = \mathbb{Z}^{b_0} \end{array}$$

   Identify ∂_{k+1} with the connecting map for the homology long exact sequence of the triple $W \supset W_0 \supset M_0$:

$$\cdots \longrightarrow H_{k+1}(W; W_0) \xrightarrow{\partial_{k+1}} H_k(W_0; M_0) \longrightarrow H_k(W; M_0) \longrightarrow \cdots$$

8. Let M be the 3-manifold with boundary obtained by taking a regular neighbourhood of $S^1 \vee S^2$ in $\mathbb{R}^3$ considered as a cobordism from the empty set to $\partial M = (S^1 \times S^1) \sqcup S^2$. Give M a handle decomposition with one 0-, one 1- and one 2-handle. Compute the chain complex $C_*(M; \mathbb{Z}[t, t^{-1}])$ and its homology.