Lecture 3 - Vector Bundles and Surgery: Exercises

Fibre Bundles

1. Using appropriate fibration sequences, prove that $\pi_i(U(n)) \cong \pi_i(U(n+1))$ for $i \leq 2n+1$. Calculate the following:

   (a) $\pi_i(SO(n))$ for $i = 0, 1, 2$ and $n \geq 1$;

   (b) $\pi_3(SO(n))$ for $n = 1, 2, 3, 4$ (you may assume $\pi_4(S^3) = \mathbb{Z}_2$);

   (c) $\pi_i(U(n))$ for $i = 1, 2$ and $n \geq 1$.

2. *(Part III - Fibre Bundles)* Calculate $[O(2n)/U(n), S^{n^2}]$.

3. *(Part III - Fibre Bundles)* For which n does there exist an embedding $\mathbb{R}P^n \times \mathbb{R}P^n \to \mathbb{R}P^{2n}$ such that the induced map $H^*(\mathbb{R}P^{2n}; \mathbb{Z}_2) \cong H^*(\mathbb{R}P^n \times \mathbb{R}P^n; \mathbb{Z}_2)$ is an isomorphism? Does there exist an immersion $\mathbb{R}P^2 \times \mathbb{R}P^2 \to \mathbb{R}^5$?

4. *(Part III - Fibre Bundles)* Let $G = O(n)$, find $H < G$ such that $G/H \cong \mathbb{R}P^{n-1}$. Show that for n odd
$$H^*(BO(n-1); \mathbb{Q}) \cong H^*(BO(n); \mathbb{Q}).$$
   Why does this fail for n even?

Characteristic Classes

5. Find characteristic classes that completely classify:

   (a) Real vector bundles over S^2;

   (b) Complex vector bundles over S^2;

   (c) Real vector bundles over S^4.

   Find 2 linearly independent real 4-plane bundles over S^4. Hence show that there are infinitely many vector bundles over S^4 with Euler number 1.

6. Recall that $\mathrm{Spin}(n)$ is the universal cover of $SO(n)$ and that
$$\mathrm{Spin}^c(n) = (\mathrm{Spin}(n) \times\ U(1))/(A, \lambda) \sim (-A, -\lambda).$$
   For M a closed manifold show:

   (a) M is orientable if and only if $w_1(\nu_M) = 0$;

   (b) M is spinnable if and only if $w_1(\nu_M) = w_2(\nu_M) = 0$;

   (c) M admits a Spin^c-structure if and only if $w_1(\nu_M) = 0$ and $w_2(\nu_M)$ is the mod 2 reduction of an integral cohomology class.

   You may use the fact that $\mathrm{Spin}(n)$ and $\mathrm{Spin}^c(n)$ are topological groups.

7. For a closed 3-manifold M, show that $w_1^2(M) = w_2(M)$.

8. *(Diarmuid Crowley - Topology of Manifolds Summer School)* For a closed $4k$-dimensional manifold M, let $\tilde{\nu} : M \to B$ be a normal $(2k-1)$-smoothing. Show:

   (a) If $\partial M \simeq S^3$ then TM is trivial;

   (b) If $\partial M = \emptyset$ and $c : M \to S^{4k}$ is the map collapsing the exterior of a small disk in M to a point then there is a vector bundle ξ over S^{4k} such that $c^*\xi \cong TM$.

   (c) If J is the 'J-homomorphism' and $S\xi$ is the suspension of ξ then $J(S\xi) = 0 \in \pi^s_{4k-1}$.

Vector Bundles

9. For $h^*(\cdot)$ a multiplicative cohomology theory define the Thom class of a vector bundle ξ over M. Define the Euler class $e_h(\xi)$ of ξ and show that ξ admits a nowhere zero section if and only if $e_h(\xi)$ vanishes.

10. Let $f : S^{n-1} \to SO(k)$ be a clutching function for the bundle E_f over S^n. For appropriate r and s, construct a class $\gamma_f \in \pi_r(S^s)$ that vanishes if and only if E_f admits a nowhere 0 section.

    For $k > n$ show that $E_f = E' \oplus \epsilon$ (you may use the fact that $\pi_r(S^s) = 0$ for $r < s$).

    Show that
    $$\pi_r(SO(s)) \to \pi_r(SO(s+1))$$
    is sujective for $r < s$.

11. (a) Calculate the total Stiefel-Whitney class of $\mathbb{R}P^n$.
    (b) Calculate the total Chern class of $\mathbb{C}P^n$.
    (c) For any positive integer d, let
    $$S_d = \left\{ [z_0 : z_1 : z_2 : z_3] \in \mathbb{C}P^3 | \sum z_i^d = 0 \right\}.$$
    Calculate the Chern classes of S_d in terms of (the pullback of) the generator of $H^*(\mathbb{C}P^3; \mathbb{Z}) = \mathbb{Z}\langle x \rangle / (x^4 = 0)$.

12. **We should be able to cook up an exercise involving using the signature theorem for 4-manifolds:** $\sigma(M) = \frac{1}{3}\langle p_1(X), [X] \rangle$.

13. For $k \geq 2$, determine the number of distinct S^k-bundles over S^2.

    For $m \geq 4$, let $S^1 \hookrightarrow M^m$ be a nullhomotopic embedding. Show that the surgery on M with respect to this embedding has effect $M' = M \# N$, where N is an m-dimensional manifold to be determined. When M is spin, show that M' is not uniquely determined.

    Show that M' is uniquely determined for M not spin (difficult!).

Knot Theory

14. Let $E_k \to S^2$ be the complex plane bundle with euler number k. Explain how to obtain the sphere-bundle $S(E_k)$ via surgery on S^3.

    Show that S^3 is the universal cover for $S(E_k)$ and describe the deck transformations.

15. *(Diarmuid Crowley - Topology of Manifolds Summer School)* Let W_k be the trace of the surgery on S^3 with effect $S(E_k)$. Verify that by gluing D^4 to the component of ∂W_k that is S^3 we obtain a space homotopic to S^2 and with intersection form $[\pm k]$.

16. Consider the Hopf link in S^3 with 0-framed components. Show directly that the two surgeries these define have combined effect S^3.

17. *(Part III - Morse Theory, 2005)* Suppose $\iota : S^1 \times D^1 \hookrightarrow T^2$ is an embedding such that $\iota(S^1 \times \{0\}$ is a (p,q)-curve i.e. represents the homology class $pa + qb$ where $a, b \in H_1(T^2;\mathbb{Z})$ are the two standard generators. Let M be the manifold obtained from T^2 by surgery with respect to this embedding. Compute $H_*(M;\mathbb{Z})$ for the case $p, q > 0$. Deduce a necessary and sufficient condition on strictly positive integers p and q for a (p,q)-curve to exist. (You may assume that a closed, oriented, smooth real surface is diffeomorphic to a surface of genus g for some $g \geq 0$.)

18. Let K be an embedded $S^1 \hookrightarrow S^3$ with a closed tubular neighbourhood $\nu K \cong S^1 \times D^2$. A *Dehn surgery* on K is the process of removing $\mathrm{int}(\nu K)$ and gluing back a copy of $S^1 \times D^2$ by any diffeomorphism

$$\phi : S^1 \times \partial D^2 \to \partial \nu K$$

of the boundary tori. Orienting K, let μ be a right-handed meridian and $\lambda \in H_1(\partial \nu K;\mathbb{Z})$ be a 0-framed copy of K pushed to the boundary of νK. A Lens space $L(p,-q)$ is defined to be the effect of Dehn surgery on the standard embedding $S^1 \hookrightarrow S^3$ with ϕ such that

$$\phi_*([\partial D^2]) = p\mu + q\lambda.$$

(a) Show $L(\pm 2, 1) \simeq \mathbb{R}P^3$, $L(\pm 1, 1) = L(p, 0) = S^3$.

(b) Prove the 'slam dunk' - that the combined effect of the two surgeries on the Hopf link in S^3 with framings m and n on the respective components is the Lens space $L(1-mn, n)$. Hence show that any Lens space is null-cobordant (Hint: it may help to prove that $L(p,-q) = L(-p,q)$ so that we can unambiguously consider the Dehn surgery generating the space as 'p/q-surgery' on the embedded S^1).

19. **Should be a nice exercise about plumbing somewhere in here... How to interpret plumbing as surgery?**

Here's a good exercise but I don't know where to put it:
(Part III - Algebraic Topology, 2005) Show that

- $K(G,n) \times K(H,n) \simeq K(G \times H, n)$.
- Describe a $K(\mathbb{Z}_p, 1)$. Calculate the rings $H^*(K(\mathbb{Z}_p;\mathbb{Z}_p)$ and $H^*(K(\mathbb{Z}_p \times \mathbb{Z}_p, 1);\mathbb{Z}_p)$.
- Let M^m be a cell complex and X be the result of attaching a single $(n+1)$-cell and finitely many i-cells to M (for $i \geq n+2$). Show that $H^{n+1}(X;\mathbb{Z}_p) = 0$ or $\mathbb{Z}_p$.

Let G act on S^n for $n > 1$ and $M = S^n/G$. By adding cells to kill $\pi_i(M)$ for $i \geq n$, show that $G \neq \mathbb{Z}_p \times \mathbb{Z}_p$. (You may assume homotopy groups of spheres are finitely generated.)