

Whitehead Torsion I

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X, Y finite CW complexes.

Let $f: X \rightarrow Y$ be a homotopy equivalence. What can we say about f ?
Can we improve f , within its homotopy class, to become something stronger e.g. a homeomorphism?

In general we cannot: there is a topological invariant "Reidemeister torsion" taking values in \mathbb{C}^* for instance.
One can show that $L(7,1) \simeq L(7,2)$ but $L(7,1) \not\simeq L(7,3)$.
Here $L(m,1)$ is the lens space

$$\begin{aligned} L(m,1) &= S^3 / \mathbb{Z}_m \\ &= \{ (a, b) \in \mathbb{C} \times \mathbb{C} : |a|^2 + |b|^2 = 1 \} / (a, b) \sim (\xi a, \xi^m b) \\ &\text{with } \xi = e^{\frac{2\pi i}{m}} \text{ being an } m^{\text{th}} \text{ root of unity.} \end{aligned}$$

We can lift a homotopy equivalence $f: X \rightarrow Y$ to a homotopy equivalence $\tilde{f}: \tilde{X} \rightarrow \tilde{Y}$ of universal covers. Then we may identify $\pi_1(Y) \cong \pi_1(X)$ as the group of deck transformations.

$$\sigma: (\tilde{X}, \tilde{Y}) \longrightarrow (\tilde{X}, \tilde{Y})$$

Can be assumed to be cellular. Then $\sigma_*: C_*^{\text{cell}}(\tilde{X}, \tilde{Y}) \rightarrow C_*^{\text{cell}}(\tilde{X}, \tilde{Y})$ is a chain map of cellular chain complexes. Then $C_*^{\text{cell}}(\tilde{X}, \tilde{Y})$ is a chain complex of $f.g.$ free $\mathbb{Z}[\pi_1(X)] = \mathbb{Z}[\pi_1(Y)]$ modules. Since f is a homotopy equivalence $H_*^{\text{cell}}(\tilde{X}, \tilde{Y}) = 0$.

geometric
 Take a basis of $C_r(X, Y)$ and then lift to a basis of $C_r(\tilde{X}, Y)$
 but a choice of lift has been made.

Algebra

Lemma A chain complex C is contractible ($1_C \cong 0_C: C \rightarrow C$)
 if and only if C is acyclic and

$$0 \rightarrow \ker(d_{n+1}) \xrightarrow{i} C_{n+1} \xrightarrow{d_{n+1}} \operatorname{im}(d_{n+1}) \rightarrow 0$$

splits for all n .

Proof: \Rightarrow Any contractible sequence is acyclic.

~~Since $H_*(C) = 0$ then~~

Let s be a chain homotopy between 1_C and 0_C .

Then $sd + ds = 1$ so $\operatorname{im}(d_{n+1}) \xrightarrow{sl} C_{n+1}$ splits d_{n+1} .

\Leftarrow If we have a splitting h then

$$C_{n+1} \cong \ker(d_{n+1}) \oplus \operatorname{im}(d_{n+1})$$

and set $S_{n+1} = h \oplus 0$.

Corollary: A chain complex of f.g. projective modules
 bounded below is acyclic iff it is contractible.

Proof: $\ker d_0 \cong C_0 \Rightarrow \ker d_0$ proj.

Then recursively $\ker d_n$ proj \Rightarrow SES always splits.

Lemma If C is acyclic with chain contraction $s: 0 \cong 1: C \rightarrow C$ then

$$s+d = \begin{pmatrix} d & 0 & 0 & \dots & \dots \\ s & d & 0 & \dots & \dots \\ 0 & s & d & \dots & \dots \\ 0 & 0 & s & \dots & \dots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix} : C_{\text{odd}} = \bigoplus_{n \in \mathbb{Z}} C_{2n+1} \longrightarrow C_{\text{even}} = \bigoplus_{n \in \mathbb{Z}} C_{2n}$$

is an isomorphism of chain complexes. \mathbb{R} -modules.

Proof: Inverse: $\begin{pmatrix} s & d & 0 & \dots & \dots \\ 0 & s & d & \dots & \dots \\ 0 & 0 & s & \dots & \dots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix} = d+s$

$$(s+d)(d+s) = \begin{pmatrix} s^2 & 1 & & & \\ & s^2 & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & \ddots \end{pmatrix} = I + N$$

\uparrow
nilpotent

$\Rightarrow (s+d)(d+s)$ is an isomorphism
Similarly $(d+s)(s+d)$ is an isomorphism.

Remark: If \mathbb{R} is commutative then matrices have determinants.

Suppose I choose a preferred basis of C . Then the Reidemeister torsion can be defined as

$$\Delta(C) = \det(s+d: C_{\text{odd}} \longrightarrow C_{\text{even}}) \in \mathbb{R}^{\times}$$

Definition $GL(\mathbb{R}) = \varinjlim GL(n, \mathbb{R})$ where

$$GL(n, \mathbb{R}) \hookrightarrow GL(n+1, \mathbb{R})$$

$$A \longmapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$$

is the infinite general linear group. In general, this is not abelian.

Let $E_n(\mathbb{R}) \triangleleft GL(n, \mathbb{R})$ be the normal subgroup generated by $(I + r E_{ij})$ with $(G_{ij})_{r \in \mathbb{R}} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{otherwise} \end{cases}$
 with direct limit $E(\mathbb{R}) = \varinjlim E_n(\mathbb{R})$.

Claim: (Whitehead Lemma)

$$E(\mathbb{R}) = [GL(\mathbb{R}), GL(\mathbb{R})]$$

Proof: \subseteq Any elementary matrix is a commutator
 $I + r E_{ij} = [I + r E_{ik}, I + r E_{kj}]$

\supseteq We need to show $[A, B] \in GL(\mathbb{R})$ is a product of elementary matrices.

$$\begin{aligned}
 [A, B] &= ABA^{-1}B^{-1} \in GL(n, \mathbb{R}) \\
 &= \begin{pmatrix} ABA^{-1}B^{-1} & 0 \\ 0 & I_n \end{pmatrix} \in GL(2n, \mathbb{R}) \\
 &= \begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix} \begin{pmatrix} B & 0 \\ 0 & B^{-1} \end{pmatrix} \begin{pmatrix} (BA)^{-1} & 0 \\ 0 & BA \end{pmatrix}
 \end{aligned}$$

} inside $GL(\mathbb{R})$

So only need to show matrices $\begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix}$ are products of elementaries

$$\begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix} = \begin{pmatrix} I & c \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ -c^{-1} & I \end{pmatrix} \begin{pmatrix} I & c \\ 0 & I \end{pmatrix} \begin{pmatrix} 0 & -I \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ I & I \end{pmatrix} \begin{pmatrix} I & -I \\ 0 & I \end{pmatrix}$$

and $\begin{pmatrix} I & X \\ 0 & I \end{pmatrix} = \prod_{i=1}^n \prod_{j=1}^n (I_{2n} + x_{ij} E_{ij})$

~~Lemma~~ Let $G < R^X$ and set $E_G \triangleq GL(R)$ generated by $E(R)$ and matrices $\begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & g_i & \\ & & & \ddots \\ 0 & & & & 1 \end{pmatrix}$ for $g \in G$.

Definition: $\pi_G(R) = GL(R)$
Eg.

Special cases: $G = S13$ then $\pi_G(R) = \pi_1(R) = \frac{GL(R)}{E(R)} = \frac{GL(R)}{(GL(R), GL(R))}$

$G = S\pm 1$ then $\pi_G(R) = \tilde{\pi}_1(R) = \frac{\pi_1(R)}{\pm 1}$

$R = \mathbb{Z}G$, $\pi_G(\mathbb{Z}G) = Wh(G) = \frac{\pi_1(R)}{S\pm G}$ is the

Whitehead group.

Definition $\tau_G: GL(R) \longrightarrow \pi_G(R)$
 $A \longmapsto [A]$

Whitehead torsion $\tau: GL(R) \longrightarrow Wh(G)$ $R = \mathbb{Z}[G]$
 $A \longmapsto [A]$

Example $wh(S^1)$

• $G = S^1$ $\frac{H_1(\mathbb{Z})}{S^1} = H_1(\mathbb{Z}) = 0.$

• $G = \mathbb{Z}$ $\frac{H_1(\mathbb{Z}(\mathbb{Z}))}{\pm \mathbb{Z}} = 0$

\parallel
 $wh(\mathbb{Z})$

• $G = \mathbb{Z}^n$, $wh(\mathbb{Z}^n) = 0$ (Kass-Heller-Swan).

• $G = \mathbb{Z}_2$, $wh(\mathbb{Z}_2) = 0$
 $wh(\mathbb{Z}_3) = 0.$

G commutative $\Rightarrow \mathbb{Z}G$ is commutative
 \Rightarrow we have determinants

$$0 \rightarrow St_1(\mathbb{Z}G) \rightarrow wh(G) \rightarrow \frac{(\mathbb{Z}G)^*}{S^1 G} \rightarrow 0.$$

$$wh(G) = St_1(\mathbb{Z}G) \oplus \frac{(\mathbb{Z}G)^*}{S^1 G}.$$

Example If $G = \mathbb{Z}_5$ then $\mathbb{Z}(\mathbb{Z}_5)$ has units $x \in \mathbb{Z}(\mathbb{Z}_5)$
s.t. $x \notin \mathbb{Z}_5$ so $wh(\mathbb{Z}_5) \neq 0.$

Def Let C be ^{a based} an acyclic chain complex of f.g. projective $\mathbb{Z}G$ -modules bounded below or above. Then

$$\tau(C) = \tau(S^{+d}: C_{\text{odd}} \rightarrow C_{\text{even}}) \in wh(G)$$

Well-defined? • choice of chain contraction - independent.
• choice of basis.

$f: X \rightarrow Y$
 $f_* \cdot C(\tilde{X}) \xrightarrow{\cong} C(\tilde{Y}) \Rightarrow C_*(f) = \text{cone}(f_*)$ is acyclic + based.

Addenda: Let C be a based chain complex of $f.g.$ projective modules over $R = \mathbb{Z}G$. If C is acyclic then there is a chain contraction $s: C_n \rightarrow C_{n+1}$. Then $\text{std}: C_{\text{odd}} \rightarrow C_{\text{even}}$ is an isomorphism of $\mathbb{Z}G$ -modules. Then there exists $i: C_{\text{even}} \rightarrow C_{\text{odd}}$ which is an inverse to std . Then $\tau(i) = [i \circ \text{std}] \in \text{Wh}(G) = \frac{\pi_1(\mathbb{Z}G)}{\{\pm g\}}$.

Well defined? Take i, i' and $\tau(i \circ \text{std}) = \tau(i \circ i'^{-1}) + \tau(i' \circ \text{std})$
 $= \tau(i' \circ \text{std})$
 where $\tau(i \circ i'^{-1})$ is a permutation matrix of basis elements of C_{odd} .

Example (before) $\text{Wh}(\mathbb{Z}_5) = \text{Stk}(\mathbb{Z}G) \oplus \frac{(\mathbb{Z}G)^*}{\{\pm g\}} = \frac{(\mathbb{Z}G)^*}{\{\pm g\}} = \mathbb{Z}$ $G = \mathbb{Z}_5$

A non-trivial unit is $(1-t-t^4)$ since $(1-t^2-t^4)(1-t^2-t^3) = 1$.

Theorem (Bass-Milnor-Serre)

$\text{Wh}(\mathbb{Z}_n) = \mathbb{Z}^k$, $k = \left[\frac{n}{2} \right] + 1 - d(n)$, $d(n) = \#$ of positive divisors of n .

