

Whitehead Torsion II

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GEOMETRY

X, Y finite CW complexes. Original question: if $f: X \rightarrow Y$ is a homotopy equivalence can we improve f within its homotopy class to be a homeomorphism (or more generally something stronger than a homotopy equivalence)?

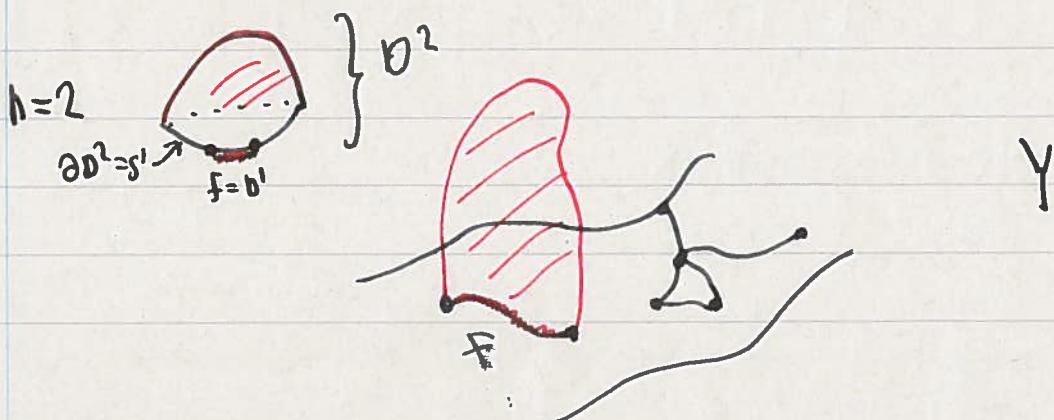
Theorem (Cellular Approximation Theorem)

Any map of CW pairs $f: (X_1, Y_1) \rightarrow (X_2, Y_2)$ is homotopic rel (Y_1) to a cellular map.

Let's look at deformation retracts $f: X \rightarrow Y$ (justification blends to why this is equivalent).

Question: If $f: X \rightarrow Y$ is a deformation retract, is X "just Y with some cells glued on"?

Definition X is an elementary expansion of Y , $X \xrightarrow{g} Y$, $Y \xleftarrow{f} X$, if $X = Y \cup D^n$ where f attaches D^n by part of its boundary, i.e. we glue $F = D^{n+1} \cap \partial D^n$, $f: F \rightarrow Y^{n-1}$ such that $\partial F \subseteq Y^{n-1}$.



$X = X_0 \xrightarrow{e} X \xrightarrow{e'} Y, \quad X \xrightarrow{f} Y \text{ or } Y \xrightarrow{g} X$
 A combination of \xrightarrow{e} and \xrightarrow{f} or \xrightarrow{g}

There is always some Z s.t. $X \xrightarrow{f} Z \xrightarrow{g} Y$

Def: $f: X \rightarrow Y$ is a simple homotopy equivalence (S.H.e)
 if f is homotopic to a composite $g: X \rightarrow Y$ of a
 sequence of elementary expansions and contractions.

$$\begin{aligned} \text{Recall } \tau(X, Y) &= \tau(C_*(\tilde{X}, \tilde{Y})) \\ &= \tau(\text{cone}(f_*: C_*(\tilde{X}) \rightarrow C_*(\tilde{Y}))). \end{aligned}$$

Claim: If $X \wedge X' \text{ (rel } Y)$ then $\tau(X, Y) = \tau(X', Y)$.

Proof: If $X \hookrightarrow X'$ then $X' = X \cup \{e^{n-1}\} \cup \{e^n\}$ where
 $e^{n-1} = \partial e^n \setminus \text{int}(e^n)$.
 $e^n = D^n$.

Choose an orientation for $[e^n], [e^{n-1}] \in C_*(\tilde{X}, \tilde{Y})$
 Then $\partial[e^n] = [e^{n-1}] + c$ where $c \in C_{n-1}(\tilde{X}, \tilde{Y})$.

Via $s: C_n(\tilde{X}, \tilde{Y}) \rightarrow C_{n+1}(\tilde{X}, \tilde{Y})$ extends to (\tilde{X}', \tilde{Y})

$$s[e^{n-1}] = [e^n] - sc, \quad s[e^n] = 0.$$

Call the new contraction s' .

New matrix

$n \text{ odd}$	$n \text{ even}$
$\begin{matrix} & \begin{matrix} \text{(even} & [e^{n-1}] \\ \text{(odd} & \text{dts} \\ [e^n] & 0 \end{matrix} \end{matrix}$	$\begin{matrix} & \begin{matrix} \text{(even} & [e^n] \\ \text{(odd} & \text{dts} \\ [e^{n-1}] & 0 \end{matrix} \end{matrix}$
$-sc$	c
1	1

Then $[dts'] = [dts] \in \pi_1(\mathbb{Z}G)$.

Claim: $\tau(X, Y)$ independent of orientations of cells.

- lifts of cells to universal cover.
- base point / identification of $\pi_1 = \text{cov}(Y \rightarrow Y)$.

Proof: Changing the lift or orientation of a cell in $X - Y$ changes the matrix by multiplication by

$$\begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & \pm g & \\ & & & \ddots \end{pmatrix}.$$

Lemma: $f: G \rightarrow G$ induces $f = \text{id}: W(G) \rightarrow W(G)$.
 $x \mapsto gxg^{-1}$

Pf: $f: \mathbb{Z}G \rightarrow \mathbb{Z}G$; $\sum n_{x,y} x \mapsto \sum n_{x,y} gxg^{-1}$
 $g \left(\sum n_{x,y} x \right) g^{-1}$

$$\Rightarrow f: GL(N; \mathbb{Z}G) \longrightarrow GL(N; \mathbb{Z}G)$$

$$A \longmapsto (g^{-1}A|g^{-1})$$

But $Wh(G)$ is abelian $\Rightarrow f = id$ on $Wh(G)$.

We have shown: if $X \wedge Y$ then $\tau(X, Y) = 0$.

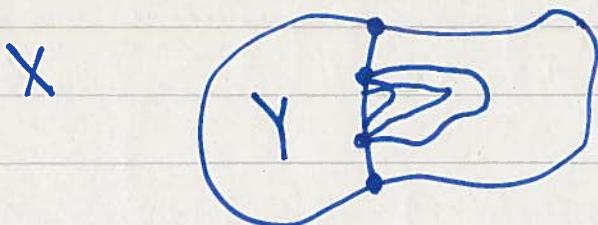
Conversely?

Proposition Whitehead Cell Trading (a.k.a folding) Lemma.

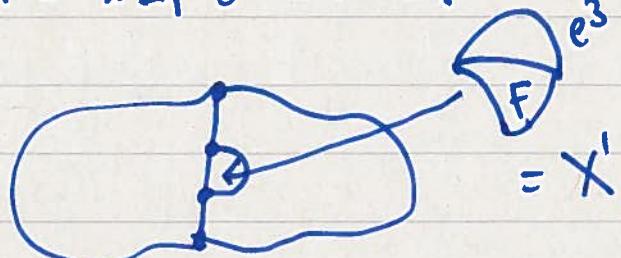
(X, Y) finite CW pair with $\pi_{k+1}(X, Y) = 0$ for $k \leq n$
then $X \wedge \bar{X}$ may s.t. $\dim(\bar{X} - Y) \geq n+1$.

Proof: WLOG $X = Y \cup \{e_k^{att}\} \cup \{e_j^{att}\} \cup \dots \cup \{e_l^N\}$.

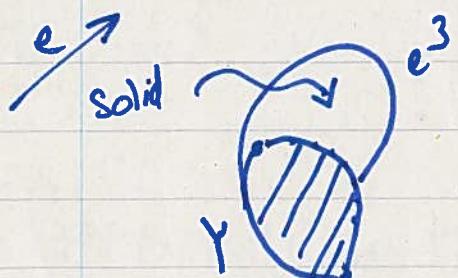
Take e_i , $i \leq N$, and trade it to an e^{i+2} .
 $\pi_i(X, Y) = 0 \Rightarrow$ homotopy $\Phi: (D^i, S^{i-1}) \times I \longrightarrow (X, Y)$.



$$\Phi: (D^i \times I, S^{i-1} \times S^{i-1}) \longrightarrow (X, Y)$$



Example: $X = Y \cup e^1 \cup e^2 \cup f$ trade e^1 for e^3 .



$$C := \partial e^3 \setminus \text{int}(f)$$

$C \setminus$ collapse C .



$$= Y \cup e^2 \cup e^3.$$

Remove int f from ∂e^3 to get C

$$c: C \amalg Y \longrightarrow Y$$

$$\bar{X} = X \underset{C}{\cup} Y, \quad X \wedge \bar{X} \text{ rel } Y.$$

Corollary: Any deformation retract $f: X \longrightarrow Y$ can be traded up to "simple form". i.e. $\exists \bar{X}$ s.t. $X \wedge \bar{X} \text{ rel } Y$ and $\bar{X} = Y \cup \{e_i^n\} \cup \{e_j^{n+1}\}$ \square .

Claim: If $\tau(X, Y) = 0$ then $X \wedge Y \text{ rel } Y$.

Proof: Assume $X = Y \cup \{e_i^N\} \cup e_j^{NH}$

$$\Rightarrow C_*(\tilde{x}, \tilde{y}): o \rightarrow C_{N+1}(\tilde{x}, \tilde{y}) \xrightarrow{\partial} C_N(\tilde{x}, \tilde{y}) \xrightarrow{\cong} o.$$

$$\tau([a]) = o \in \text{Wh}(\pi_1(Y)).$$

- multiply by matrices $\begin{pmatrix} 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 \end{pmatrix} \leftrightarrow$ changing choices of lift / orientation of the cells e_i^N, e_j^{NH}
- elementary matrices geometrically realised by "cell sliding."

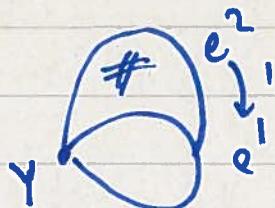
We can geometrically move

$$X \xrightarrow{\sim} X' \text{ rel } Y \text{ s.t.}$$

$$C_N(\tilde{x}', \tilde{y}') \xrightarrow[\cong]{\partial'} C_N(\tilde{x}, \tilde{y})$$

is the identity matrix.

Example.



Can collapse e^1 through e^2 as attaching map is 1.

Can use $\phi^1 = \text{id}$ to collapse to Y

$$X \xrightarrow{\sim} X' \xrightarrow{\sim} Y \text{ (rel } Y).$$

Theorem (Chapman)

If $f: X \rightarrow Y$ is a homeomorphism then $\tau(X|Y) = 0$.

