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Transversality in Algebra and Topology

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Today we will talk about the 'Higman linearization trick'. In some sense Higman began the subject of translating the geometric idea of transversality into algebra.

If we want to prove that every invertible $(a_{ij}) \in GL(N, \mathbb{Z})$ can be changed to (1) by the moves in the Whitehead group $Wh(\mathbb{Z})$.

Higman's proof shows

$$Wh(\mathbb{Z}) = 0$$

using algebra. Modern proof of this fact use topology.
Similar methods can be used to show

$$\begin{aligned} Wh(\text{finite group}) &= 0 \\ Wh(\text{infinite torsion free group}) &= 0 \end{aligned}$$

Bass, Heller, Swan (1968) showed $Wh(\mathbb{Z}^n)$ $\forall n \geq 1$ using purely algebra. It still, however, uses the vital irreducible ingredient of high-dimensional topological manifolds.

This difficult algebraic result is needed for Wall's classification of false tori and this is used % in the TOP category for manifold classification. That we need such a hard result indicates the difficulty of capturing TOP algebraically (as opposed to e.g. DIFF).

We now compare two notions of "transversality":

Higman linearization (Algebraic)

Example

$$\begin{pmatrix} a_0 + a_1 x + a_2 x^2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}$$

$$= \begin{pmatrix} a_0 + a_1 x + a_2 x^2 & 0 \\ x & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -a_2 x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_0 + a_1 x + a_2 x^2 & 0 \\ x & 1 \end{pmatrix}$$

$$= \begin{pmatrix} a_0 + a_1 x & -a_2 x \\ x & 1 \end{pmatrix}$$

So we can move a polynomial $a_0 + a_1 x + a_2 x^2$ to a linear matrix by operations permitted within the Whitehead group.

Thom transversality theorem (geometric)

Another ingredient in the early days of this subject is the Thom transversality theorem:

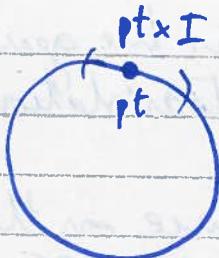
$$\begin{array}{ccc} V^n & \xrightarrow{f} & M^p \\ g & \downarrow & \downarrow \\ W^{n-q} := f^{-1}(N) & \xrightarrow{f'|_W} & N^{p-2} \end{array}$$

We can move f to a "transverse map" f' by Wpy.

$$\text{and } \nu_{W \hookrightarrow V} = (f'|_W)^* \nu_{N \hookrightarrow M}$$

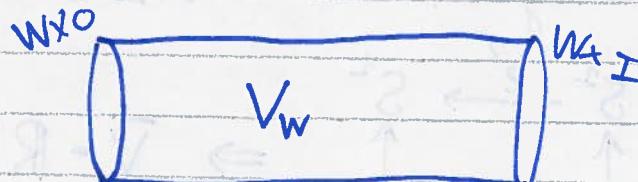
Example $M = S^1$, $N = \text{pt}$.

$$\begin{array}{ccc} V^n & \xrightarrow{f'} & M = S^1 \\ \uparrow & & \uparrow \\ W & \xrightarrow{(f')|} & \text{pt} \end{array}$$



(What is the connection
between the two?)

Consider the geometric transversality example. Cut
 V at W to get a cobordism:

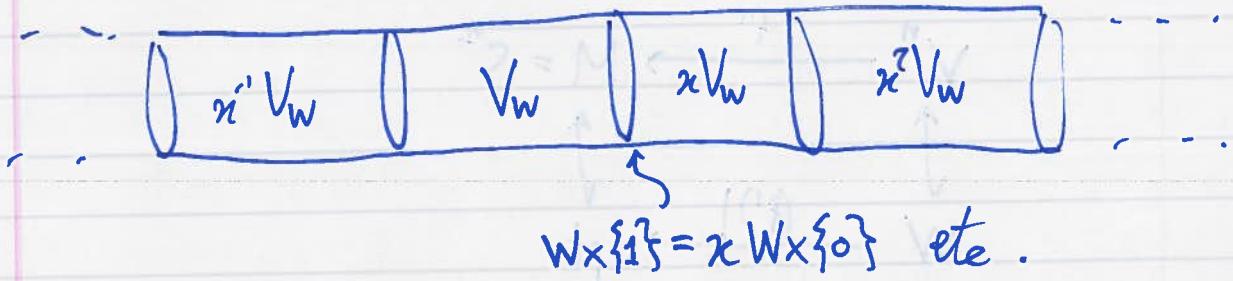


$$V_W := \text{cl}(V(W \times I)), \quad \partial V_W = W \times \{0, 1\}.$$

The cobordism is a fundamental domain for the infinite cyclic cover \bar{V} pulled back from $\mathbb{R} \rightarrow S^1$

$$\begin{array}{ccc} (f')^* \mathbb{R} = \bar{V} & \longrightarrow & \mathbb{R} \\ \downarrow & f' & \downarrow \\ V & \xrightarrow{f'} & S^1 \end{array}$$

So the infinite cyclic cover is



$x : \bar{V} \rightarrow \bar{V}$ is the generating translation for the group of covering translations.

The MV-sequence on the chain level gives a linear presentation of the $\mathbb{Z}[x, x^{-1}]$ -module chain in $C(\bar{V})$

$$0 \rightarrow C(W)[x, x^{-1}] \xrightarrow{i_0 - i_1 x} C(V_W)[x, x^{-1}] \rightarrow C(\bar{V}) \rightarrow 0$$

Where $i_0 : W x \{0\} \hookrightarrow V_W$ are the inclusions of the ends.
 $i_1 : W x \{1\} \hookrightarrow V_W$

Note the point is that $C(W)$ and $C(V_W)$ are defined over \mathbb{Z} but $C(\bar{V})$ is defined over $\mathbb{Z}[x, x^{-1}]$.

Example $V = S^1 \xrightarrow{x^2} S^1$
 $\uparrow \qquad \downarrow$
 $W = \{pt\} \cup \{pt'\} \rightarrow \{pt'\}$ $\Rightarrow \bar{V} = RUR$.

~~Check~~ $V_W = \bullet \dots \bullet \dots \bullet \dots \Rightarrow i_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} : \mathbb{Z}^2 \rightarrow \mathbb{Z}^2$
 $i_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} : \mathbb{Z}^2 \rightarrow \mathbb{Z}^2$

So $C(\bar{V})$: $0 \rightarrow \mathbb{Z}[x, x^{-1}] \xrightarrow{1-x^2} \mathbb{Z}[x, x^{-1}] \rightarrow 0$

But MV is: $0 \rightarrow \mathbb{Z} \oplus \mathbb{Z}[x, x^{-1}] \xrightarrow{\begin{pmatrix} 1 & x \\ x & 1 \end{pmatrix}} \mathbb{Z} \oplus \mathbb{Z}[x, x^{-1}] \rightarrow C(\bar{V}) \rightarrow 0$.

and, the linearization trick is

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1-x^2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} = \begin{pmatrix} 1 & x \\ x & 1 \end{pmatrix}$$

Which shows Higman agrees with the transversal MV presentation & breaks down

$\mathbb{Z}^2 : S^1 \rightarrow S^1$ was a very simple example. In general we can reconcile the 2 concepts of transversality by "adding an extra dimension".

Let V be a finite CW cx with a map $f: V \rightarrow S^1$ and a generator for the covering translations $t: \bar{V} \rightarrow \bar{V}$. The mapping torus

$$T(t) := \bar{V} \times [0, 1] / \{(v, 0) \sim (tv, 1) | v \in V\}$$

is an infinite CW cx with 2 key properties

$$\begin{array}{ccc} \rightarrow & \mathbb{R} \\ \bar{p} & \downarrow p \\ \rightarrow & S^1 \end{array}$$

(i) The projection $T(t) \rightarrow V; (v, s) \mapsto \bar{p}(v)$ is a htpy equivalence. The fibers are copies of \mathbb{R} .

(ii) The pullback infinite cyclic cover $\overline{T(t)}$

$$\begin{array}{ccccc} \overline{T(t)} & \longrightarrow & \bar{V} & \longrightarrow & \mathbb{R} \\ \downarrow & & \downarrow & & \downarrow \\ T(t) & \xrightarrow{\cong} & V & \longrightarrow & S^1 \end{array}$$

is just $\widehat{T}(t) = R \times \bar{V}$ with generating covering translation

$$R \times \bar{V} \rightarrow R \times \bar{V}$$

$$(r, v) \mapsto (r+1, t(v))$$

$$T(t) = R \times_{\mathbb{Z}} \bar{V} \quad i_0: W = \bar{V} \times \{0\} \rightarrow \bar{V} \times [0,1]; v \mapsto (v, 0)$$

$$i_1: W = \bar{V} \times \{1\} \rightarrow \bar{V} \times [0,1]; v \mapsto (tv, 1)$$

$\bar{V} \times [0,1]$ is a canonical fundamental domain

It is canonical in the sense that, as opposed to the transversal arguments from before, NO CHOICE has been made in defining this. It simply comes from the $[0,1]$ in $T(t)$. The price we paid for this is non-compactness.

Theorem For any finite CW ex V and a map $f: V \rightarrow S^1$ there exist subcomplexes $X \subset \bar{V}$ s.t.

$$\bigcup_{k=-\infty}^{\infty} t^k X = \bar{V}$$

$$\text{Then } X \cap t^{-1}X \quad t(X \cap t^{-1}X) = tX \cap X$$

$$\boxed{\begin{matrix} f & \rightarrow & X \times I \\ i_0 & & \end{matrix}}$$

is a finite fundamental domain for the infinite cyclic cover

$$V(X) := \frac{X \times I}{\{(y, 0) n(ty, 1) \mid y \in X \cap t^{-1}X\}}$$

Moreover $V(X) \hookrightarrow T(t) \xrightarrow{\sim} V$ is a htpy equivalence.

simple

Remark Any ~~not~~ htpy equivalence with contractible point inverses is a simple htpy equivalence.

Next time: Proof!

staple

option in $V \in \mathbb{E}(AT) \rightarrow (XV)$ ~~remove~~
~~and remove~~

new W diff.

many alternatives after removing gold are not closed
relationships exist between them so it remains

2019 : next tool