

## Transversality In Algebra and Topology II

Chain complexes over a polynomial extension ring  $A[z, z^{-1}] = A(\mathbb{Z})$ .  
(also works for  $A[z]$ ).

Want to describe modules, quadratic forms etc over  $A[z, z^{-1}]$   
in terms of corresponding objects over  $A$ , using fundamental domains.

A CW complex  $X$  over  $S^1$  is a map  $X \rightarrow S^1$ . Let  $\tilde{X}$  be the  
pullback cover

$$\begin{array}{ccc} \tilde{X} & \longrightarrow & \mathbb{R} \\ \downarrow & & \downarrow \\ X & \longrightarrow & S^1 \end{array}$$

The cellular chain complex  $C(\tilde{X})$  is a free  $\mathbb{Z}[z, z^{-1}]$ -module  
chain complex. (We can equally well work with  $C(X)$ ,  
 $\tilde{X}$  the universal cover of  $X$  which is a free  $\mathbb{Z}[\pi_1, \lambda]$ -module  
chain complex).

Want to describe CW complexes over  $S^1$  in terms of  
"simply-connected" CW complexes using fundamental domains.

induction

$$\begin{aligned} h: A &\longrightarrow A[\mathbb{Z}, \mathbb{Z}^{-1}] \\ h_!: \text{f.g. free } A\text{-modules} &\longrightarrow \text{f.g. free } A[\mathbb{Z}, \mathbb{Z}^{-1}]\text{-modules} \end{aligned}$$

inclusion

$$M \xrightarrow{\quad} h_! M = \underset{A}{A[\mathbb{Z}, \mathbb{Z}^{-1}] \otimes M}$$

restriction

$$\begin{aligned} h^!: A[\mathbb{Z}, \mathbb{Z}^{-1}]\text{-modules} &\longrightarrow A\text{-modules} \\ N &\longmapsto h^! N = N \text{ with } A\text{-action by } A \subset A[\mathbb{Z}, \mathbb{Z}^{-1}] \\ A \xrightarrow{h_!} h_! A = A[\mathbb{Z}, \mathbb{Z}^{-1}] &\xrightarrow{h^!} h^! A[\mathbb{Z}, \mathbb{Z}^{-1}] = \bigoplus_{j=-\infty}^{\infty} \mathbb{Z}^j A \end{aligned}$$

=  $M[\mathbb{Z}, \mathbb{Z}^{-1}]$   
infinitely generated

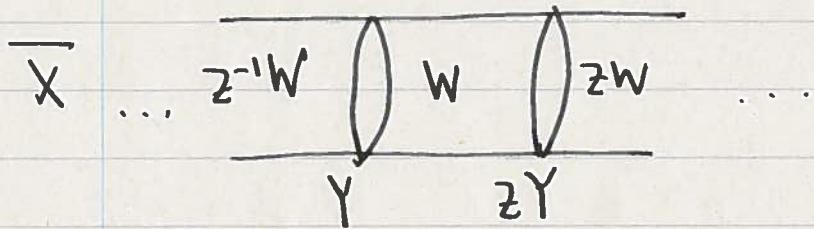
$h^! C(\bar{X}) = C(\bar{X})$  as a  $\mathbb{Z}$ -module chain complex without the  $\mathbb{Z}[\mathbb{Z}, \mathbb{Z}^{-1}]$  module structure

$h_! C(Y) = C(Y)[\mathbb{Z}, \mathbb{Z}^{-1}] = C(Y \times \mathbb{Z}) = \mathbb{Z}[\mathbb{Z}, \mathbb{Z}^{-1}]$ -module chain complex of the trivial  $\mathbb{Z}$ -cores

$$\begin{array}{ccc} \bar{Y} = Y \times \mathbb{Z} & \longrightarrow & \mathbb{R} \\ \downarrow & & \downarrow \\ Y & \xrightarrow{*} & S \end{array}$$

The functors  $h_!$  and  $h^!$  are adjoint:  $\text{Hom}_{A[\mathbb{Z}, \mathbb{Z}^{-1}]}(h_! M, N) = \underset{A}{\text{Hom}}(M, h^! N)$

A fundamental domain of an infinite cyclic cover  $\bar{X}$  of an infinite cyclic cover  $\bar{X}$  of a CW complex  $X$  with a map  $X \rightarrow S^1$  is a subcomplex  $W \subset \bar{X}$  such that  $\bigcap_{j=-\infty}^{\infty} W \cap W = Y$  is disjoint from  $ZY = W \cap ZW$  and  $\bar{X} = \bigcup_{j=-\infty}^{\infty} Z^j W$

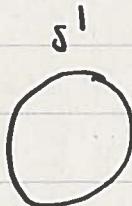
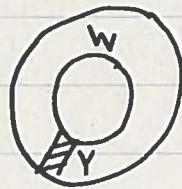


An infinite cyclic cover  $\overline{X}$  of a manifold  $X$  has fundamental domain by manifold transversality: make the map  
 $f: X' \rightarrow S^1$  transverse regular at  $* \in S^1$ .

$$\begin{array}{ccc} & \leftarrow & \\ \downarrow & & \downarrow \\ Y = f^{-1}(*) & \longrightarrow & * \end{array}$$

$$W = \overline{X - YxI}$$

$$\partial W = Y \sqcup zY$$



Def: A <sup>finite</sup> subfundamental domain for an infinite cyclic cover  $\overline{X}$  of a CW complex  $X$  is a subcomplex  $W \subseteq \overline{X}$ , such that

$$(ii) \quad \bigcup_{j=-\infty}^{\infty} z^j W = \overline{X} \quad \text{with } Y = W \cap z^{-1}W$$

(iii)  $\frac{W \times I}{\begin{matrix} Y \sim zY \\ \parallel \\ X' \end{matrix}} \xrightarrow{\text{Proj}} X$  is a hereditary homotopy equivalence (for any  $U \subseteq X$  open,  
 $p|: p^{-1}(U) \rightarrow U$  is also a homotopy equivalence).

[For finite  $X'$  (e.g. if  $W$  is finite) then hereditary  $\Leftrightarrow$  simple and onto]  
 $\xrightarrow{\text{homotopy equivalence}}$   $\xrightarrow{\text{homotopy equivalence}}$

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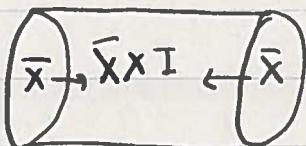
$$y \mapsto (y, 0) \quad (z, 1) \leftrightarrow zy$$

$\bar{X}$  has fundamental domain  $(WxI, Y, Zy)$

The universal subfundamental domain  $W = \bar{X}$  with  $W \cap z^{-1}W = \bar{X} \cap \bar{X} = \bar{X}$

$$X' = \frac{\bar{X} \times I}{(S^1, 0) \sim (z\bar{x}, 1)} = T(z\bar{x})$$

$\mathbb{R} \rightarrow X' \xrightarrow{p} X$  is a fibre bundle  $p^{-1}(0) \cong \mathbb{R}$   
 $p$  is a hereditary homotopy equivalence.



$$\bar{x} \mapsto (\bar{x}, 0) \quad (z\bar{x}, 1) \leftrightarrow z\bar{x}$$

Theorem For any finite CW complex  $X$  w/ infinite cyclic cover  $\bar{X}$  there exists finite subfundamental domains  $W \subset \bar{X}$ , and  $\bar{X} = \bigcup W$

$W \subset \bar{X}$   
finite  
subfundamental  
domain

Proof:  $X = \bigcup_{e_0} e^0 \cup \bigcup_{e_1} e^1 \cup \dots \cup \bigcup_{e_N} e^N$

$$\bar{X} = \bigcup_{i=-\infty}^{\infty} \left( \bigcup_{c_0} z^{i-\bar{e}^0} \cup \bigcup_{c_1} z^{i-\bar{e}^1} \cup \dots \cup \bigcup_{c_N} z^{i-\bar{e}^N} \right)$$

$\bar{e}^j \subset \bar{X}$  is a lift of  $e^j \subset X$  with  $p^{-1}(e^j) = \prod_{i=-\infty}^{\infty} z^i e^j$

We will build  $W(\bar{X})$  from the top down.

STEP N:  $W_N = \bigcup_{i=-\infty}^{\infty} \dots$

$W_N \supset W_{N-1} \supset \dots \supset W_0$  = finite subfundamental domain.

$$W_N = \bigcup_{n=0}^{N-1} \left( \bigcup_{i=-\infty}^{\infty} z^i \bar{e}^n \right) \cup \bigcup_{c_N} \bar{e}^N$$

$$X = S^1 \times S^2 \longrightarrow S^1, \quad S^1 \times S^2 = e^0 \cup e^1 \cup e^2$$

$$\bar{X} = \overline{e^2} \cup \overline{ze^2} \cup \overline{z^2e^2}$$

$$W_2 = \dots \bigcup_{R} \overline{e^2} \dots \bigcup_{a \leq j \leq b} \left( \bigcup_{[a_1, b_1] \subseteq R} z^j \bar{e}^{N-1} \right) \quad R = \bigcup_{[a_1, b_1]} [a_1, b_1]$$

$$W_1 = \overline{e^0} \cup \overline{e^1} \cup \overline{e^2} \quad //$$

$$W_{N-1} = \bigcup_{n=0}^{N-2} \left( \bigcup_{i=-\infty}^{\infty} z^i \bar{e}^n \right) \cup \left( \begin{array}{l} \text{as many } (N-1)\text{-cells} \\ \text{in } \bar{X} \text{ as required for} \\ N\text{-cells to be} \\ \text{attached} \end{array} \right) \cup \bigcup_{c_N} \overline{e^N}$$

:

This is finite

$W_0$  = finite subfundamental domain.

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$$\begin{array}{ccc}
 & \downarrow \mathbb{R} & \\
 X' = \frac{WxI}{Y_{x0} \sim ZY_{x1}} & \longrightarrow & T(z \cdot \bar{x} \cdot \bar{y}) \rightarrow S = I / \begin{matrix} 0 \\ 0 \end{matrix} \\
 & \searrow \begin{matrix} p' \\ \text{hereditary} \end{matrix} & \downarrow \begin{matrix} p \\ \text{hereditary} \end{matrix} \\
 & X &
 \end{array}$$

Def A Mayer-Vietoris presentation of an  $A[z_1, z_1^{-1}]$ -module chain complex  $C$  is an exact sequence of the type

$$0 \longrightarrow EC[z_1, z_1^{-1}] \xrightarrow{f \cdot zg} DC[z_1, z_1^{-1}] \longrightarrow C \longrightarrow 0$$

with  $D, E$   $A$ -module chain complexes,  $f, g : E \rightarrow D$

$$\begin{array}{ccc}
 \boxed{\begin{matrix} E \\ \xrightarrow{f} \end{matrix}} & D & \boxed{\begin{matrix} E \\ \xleftarrow{g} \end{matrix}} \\
 \begin{matrix} A[z_1, z_1^{-1}] \text{-modules} \\ \downarrow f^! \\ A \text{-modules} \end{matrix} & & \begin{matrix} A[z_2, z_2^{-1}] \text{-modules} \\ \nearrow g^! \end{matrix}
 \end{array}$$

Theorem: Every f.g. free  $A[z_1, z_1^{-1}]$ -module chain complex  $C$  has a canonical infinitely generated free  $A$ -module chain complex

$$0 \longrightarrow \pi_1 \pi_! C \xrightarrow{1 - z_1 z_2} \pi_1 \pi_! C \longrightarrow C \longrightarrow 0.$$

Inside this universal one there are lots of finite ones

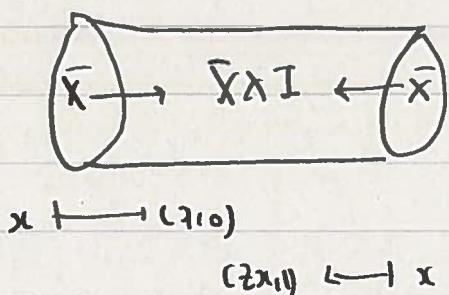
$$\begin{array}{ccccccc}
 0 & \longrightarrow & \pi_1 \pi_! C & \longrightarrow & \pi_1 \pi_! C & \longrightarrow & C \longrightarrow 0 \\
 & \cup & & & \cup & & \parallel \\
 0 & \longrightarrow & \pi_1 E & \xrightarrow{f \cdot zg} & \pi_1 D & \longrightarrow & C \longrightarrow 0
 \end{array}$$

6'

$$\begin{array}{ccc}
 \mathbb{R} & = & \mathbb{R} \\
 \downarrow & & \downarrow \\
 \bar{X} \times \mathbb{R} & \longrightarrow & T(z: \bar{X}^2) \longrightarrow S' \\
 \downarrow \pi & & \downarrow \pi \\
 \bar{X} & \longrightarrow & X \longrightarrow S'
 \end{array}$$

$$T(z: \bar{X}^2) = \frac{\bar{X} \times [0, 1]}{(z, 0) \sim (z, 1)} = \frac{\bar{X} \times \mathbb{R}}{z}$$

$$\begin{array}{ccccccc}
 0 & \longrightarrow & C(\bar{X})[z_1 z_1^{-1}] & \xrightarrow{1 - z_1 z_1^{-1}} & C((\bar{X})[z_1 z_1^{-1}]) & \longrightarrow & C(C(z)) \xrightarrow{z} C(\bar{X}) \\
 & & & & \parallel & & \pi[z_1 z_1^{-1}] \\
 & & & & C(\bar{X} \times I)[z_1 z_1^{-1}] & &
 \end{array}$$



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with  $D, E$  f.g. free  $A$ -module chain complexes.

$f_i = f \cdot g$ . free  $A$ -modules

$$C_N = f_N [z, z^{-1}]$$

$$\downarrow d = \sum d_j z^j$$

$$C_{N-1} = f_{N-1} [z, z^{-1}]$$

↓  
⋮

$$\text{Pf: } D_N = f_N \hookrightarrow C_N = f_N [z, z^{-1}]$$

$$\downarrow d$$

$$d(f_N) \subseteq \sum_{j=-a_{N-1}}^{b_{N-1}} z^j f_{N-1}$$

$$D_{N-1} = \sum_{j=a_{N-1}}^{b_{N-1}} f_{N-1} \hookrightarrow C_{N-1} = f_{N-1} [z, z^{-1}]$$

$$\downarrow$$

$$d\left(\sum_{j=a_{N-1}}^{b_{N-1}} z^j f_{N-1}\right) \subseteq \sum_{j=-a_{N-2}}^{b_{N-2}} z^j f_{N-2}$$

$$D_{N-2} = \sum_{j=a_{N-2}}^{b_{N-2}} f_{N-2} \quad C_{N-2} =$$

$$\text{Set } E = D \cap \mathbb{Z}^A$$

$$f: E \longrightarrow \mathbb{Z}; x \mapsto x$$

$$g: E \longrightarrow \mathbb{Z}; x \mapsto zx$$

$\mathbb{A}, E$  are f.g. free  $A$ -module chain complexes

$$0 \rightarrow \mathbb{A}_! E \xrightarrow{f - zg} \mathbb{A}_! \mathbb{A} \rightarrow C \rightarrow D.$$