

Algebraic And Geometric Transversality IV

How can Higman's original linearization trick

$$\begin{pmatrix} a_0 + a_1 x + a_2 x^2 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 - a_2 & a_0 + a_1 x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}$$

any ring A , $a_0, a_1, a_2 \in A$,
matrices over $A[x, x^{-1}]$.

be explained in terms of chain complexes?

$\text{tr}: A \rightarrow A[x, x^{-1}]$ inclusion

inductioⁿ: $\text{tr}_1: A\text{-modules} \rightarrow A[x, x^{-1}]\text{-modules}$

$$M \longmapsto \text{tr}_1 M = A[x, x^{-1}] \otimes_A M = \sum_{j=-\infty}^{\infty} x^j M$$

restriction $\text{tr}^!: A[x, x^{-1}]\text{-modules} \rightarrow A\text{-modules}$

$$N \longmapsto \text{tr}^! N \text{ with } A\text{-action by } A \subset A[x, x^{-1}]$$

These are adjoint functors : $\underset{A[x, x^{-1}]}{\text{Hom}}(\text{tr}_1 M, N) \cong \underset{A}{\text{Hom}}(M, \text{tr}^! N)$

Definition

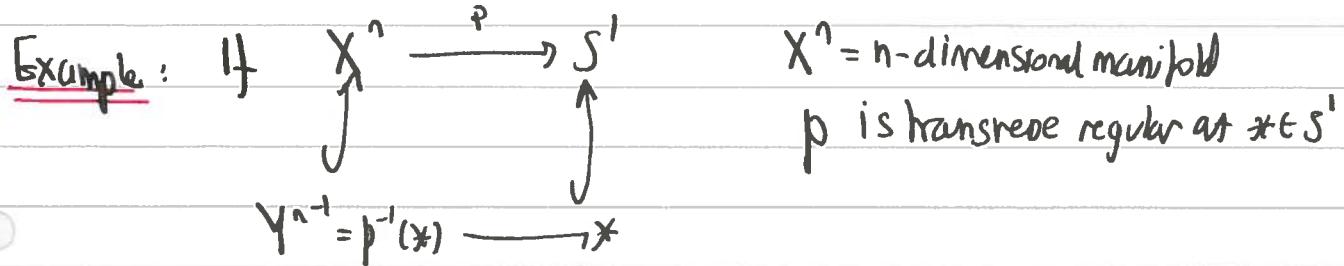
A Mayer-Vietoris presentation of an $A[x, x^{-1}]$ -module chain complex
 C is an exact sequence of $A[x, x^{-1}]$ -module chain complex

$$\Sigma: 0 \longrightarrow \text{tr}_1 E \xrightarrow{f-xg} \text{tr}_1 D \longrightarrow C \longrightarrow 0$$

$$\begin{matrix} \text{II} \\ \sum_{j=-\infty}^{\infty} x^j E \end{matrix} \qquad \begin{matrix} \text{II} \\ \sum_{j=-\infty}^{\infty} x^j D \end{matrix}$$

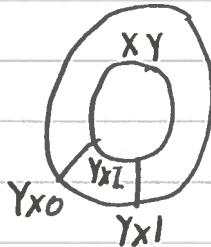
with D, E A -module chain complexes, $\text{fig: } f \rightarrow D$ A -module chain maps

$$\left(\begin{array}{c} ; \\ ; \\ ; \end{array} \right) : x^* D \xleftarrow{g} \left(\begin{array}{c} ; \\ ; \\ ; \end{array} \right) E \xrightarrow{f} \left(\begin{array}{c} ; \\ ; \\ ; \end{array} \right) D \xleftarrow{h} \left(\begin{array}{c} ; \\ ; \\ ; \end{array} \right) x^* D$$



then let $X_Y = \overline{X \setminus Y \times I}$

$$\bar{X} = \bigcup_{j=-\infty}^{\infty} \cup X_Y$$



$$\bigcap_{j=-\infty}^{\infty} \cup X_Y = \overline{X} \xrightarrow{p} \mathbb{R}$$

\cap = generating covering
 \cup = translation

$$\downarrow \quad \downarrow$$

$$X \xrightarrow{p} S^1$$

$$\begin{array}{ccccccc} & | & | & | & | & | & \\ & x^* Y & Y & X_Y & Z & x^* Y & \bar{X} \\ & | & | & | & | & | & \\ \hline & x^* Y & Y & X_Y & Z & x^* Y & \bar{X} \end{array}$$

Theorem: Every $A[\alpha_{\geq 1}]$ -module chain complex C has a canonical presentation $I - \Sigma Y^{-1}$

$$0 \longrightarrow \text{tr}_! (\text{tr}^! C) \xrightarrow{\Sigma^y + - \circ g} \text{tr}_! (\text{tr}^! C) \xrightarrow{h} \Sigma^x C \longrightarrow 0$$

$h = \text{adjoint of identity map} \in \text{Hom}_{A[x, x^{-1}]}(h^!C, C)$

HS

$$\text{Hom}_A(h^!C, h^!C) \ni 1$$

$$D = E = h^!C$$

$$f = \text{identity: } h^!C \rightarrow h^!C$$

$$g = g^{-1}: h^!C \rightarrow h^!C$$

$$0 \rightarrow A[y, y^{-1}]_{(x, x^{-1})} \xrightarrow{1 - xy^{-1}} \underbrace{(A[y, y^{-1}])_{(x, x^{-1})}}_{\substack{\pi^!C \\ \pi_1, \pi_1^!C_0}} \xrightarrow{h} A(x, x^{-1}) \rightarrow 0$$

\parallel
 C_0

$h(x) = h(y) = x.$

$xy = yx$

Theorem 2 If C is a bounded f.g. free $A(x, x^{-1})$ -module chain complex then there exist MV presentations $E(\mathcal{U}) \subset E(\omega)$ with finite

$$E(\mathcal{U}): 0 \rightarrow \pi_1 E(\mathcal{U}) \rightarrow \pi_1 D(\mathcal{U}) \rightarrow C \rightarrow 0$$

where $\mathcal{U} = \{U_0, U_{-1}, \dots, U_0\}$ a collection of finite subsets of \mathbb{R} .

\mathbb{R} is a tree with vertices $t_i, i \in \mathbb{Z}$
edges $(t_i, t_{i+1}), i \in \mathbb{Z}$.

A subtree is $U_j = [a_j, b_j], -\infty < a_j \leq b_j < \infty, a_j, b_j \in \mathbb{Z}$.

$$\begin{aligned} \varepsilon(\infty) : 0 &\longrightarrow t_1 A(y, y^{-1}) \xrightarrow{1 - xy^{-1}} t_1! A(y, y^{-1}) \rightarrow c_0 = A(x, x^{-1}) \\ \cup \\ \varepsilon(n) : 0 &\longrightarrow t_1 \left(\sum_{i=a_j+1}^{b_j} y^i A \right) \xrightarrow{1 - xy^{-1}} t_1! \left(\sum_{i=a_j}^{b_j} y^i A \right) \rightarrow c_0 = A(x, x^{-1}) \longrightarrow 0 \end{aligned}$$

↑ infinitely generated over A
↑ finitely generated over A .

edges in $U \xrightarrow{\cong}$ vertices in U ,

$$\varepsilon(\infty) = \bigcup_{\substack{U_j \subseteq \mathbb{R} \\ U_j \text{ finite} \\ \text{subtree}}} \varepsilon(U_j)$$

$$\begin{aligned} \varepsilon_{c_n}(U_n) &0 \longrightarrow 0 \longrightarrow f_n(x, x^{-1}) \xrightarrow{id} f_n(x, x^{-1}) = c_n & c_n = \sum_{i=a_j}^{b_j} d_{n,i} x^i \\ &0 \xrightarrow{y^i f_n} t_1 \left(\bigoplus_{i=a_j+1}^{b_j} y^i f_n \right) \longrightarrow t_1! \left(\bigoplus_{i=a_j}^{b_j} y^i f_n \right) \xrightarrow{(x^{a_j} \dots x^{b_j})} c_{n-1} = f_{n-1}(x, x^{-1}) \\ &\text{Define } U_{n-1} \text{ to be the smallest subtree } [a_j, b_j] \text{ capturing the coefficients in } d_n. \end{aligned}$$

$$D_1[x_1x^{-1}] = A[x_1x^{-1}] \xrightarrow{1} C_1 = A[x_1x^{-1}] \quad U_0 = S_0 \{ \}$$

$$\downarrow \begin{pmatrix} a_0 \\ a_1x \\ a_2x^2 \end{pmatrix} \quad \downarrow d = a_0 + a_1x + a_2x^2$$

$$(A \oplus y^2 A)(x_1x^{-1}) \rightarrow D_0[x_1x^{-1}] = (A \oplus Ay \oplus Ay^2)(x_1x^{-1}) \xrightarrow{y \rightarrow x} C_0 = A[x_1x^{-1}]$$

$$\begin{pmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{pmatrix}.$$

$$\begin{pmatrix} a_0 & x & 0 \\ a_1 & -1 & x \\ a_2 & 0 & -1 \end{pmatrix} \sim a_0 + a_1x + a_2x^2 \quad A[x_1x^{-1}] = A[x_1x^{-1}]$$

$$(A \oplus A)(x_1x^{-1}) \xrightarrow{\begin{pmatrix} x & 0 \\ -1 & x \\ 0 & -1 \end{pmatrix}} (A \oplus A \oplus A)(x_1x^{-1}) \xrightarrow{(1, x, x^2)} A[x_1x^{-1}]$$

$$\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix}, \quad a_0 + a_1x + a_2x^2$$

$$A[x_1x^{-1}] = A[x_1x^{-1}]$$

$$A[x_1x^{-1}] \xrightarrow{\begin{pmatrix} x \\ -1 \end{pmatrix}} (A \oplus A)(x_1x^{-1}) \xrightarrow{(1, x, x^2)} A[x_1x^{-1}]$$

$$\begin{pmatrix} a_0 \\ a_1 \end{pmatrix}, \quad a_0 + a_1x$$

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_0 & -x \\ a_1 & 1 \end{pmatrix} = \begin{pmatrix} a_0 + a_1x & 0 \\ a_1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -a_1 & 1 \end{pmatrix}$$