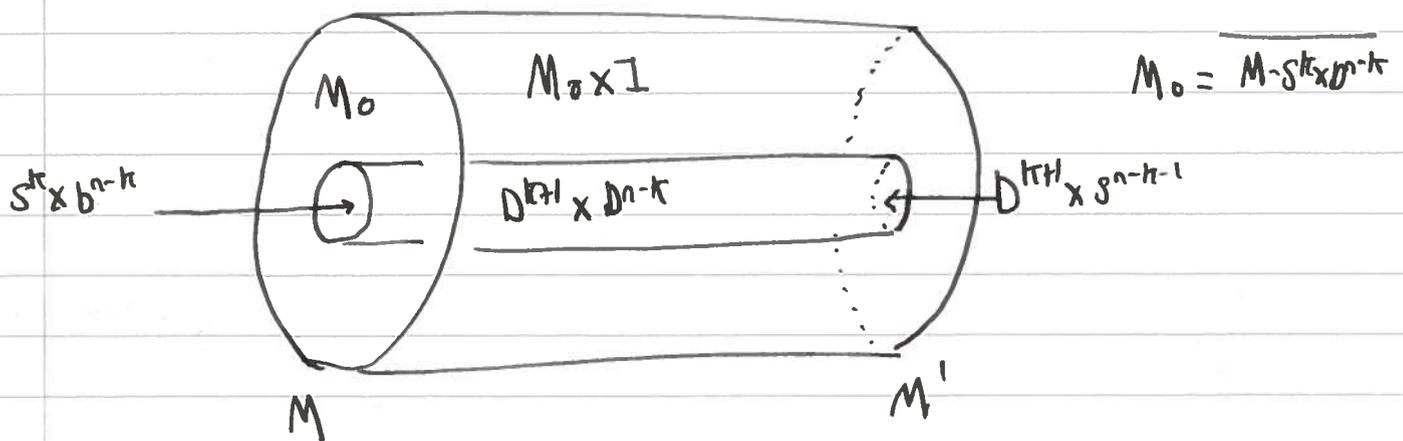


# Algebraic Surgery

The motivation for the theory of algebraic surgery comes from geometric surgery theory.

Geometric surgery: The input is an  $n$ -dimensional manifold  $M^n$  and an embedding  $S^k \times D^{n-k} \subset M^n$ ,  $-1 \leq k \leq n$ . The output is a new  $n$ -dimensional manifold  $M'^n := \overline{M - S^k \times D^{n-k}} \cup D^{k+1} \times S^{n-k-1}$ .

This is a  $\pi$ -surgery. The trace is a cobordism  $S^k \times S^{n-k-1}$  between  $M$  and  $M'$ :



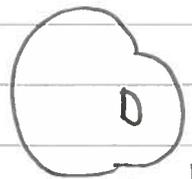
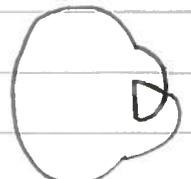
Every cobordism of manifolds is a union of traces of surgeries (Milnor via Morse Theory)



$$-1 \leq k_0 \leq k_1 \leq \dots \leq k_n \leq n$$

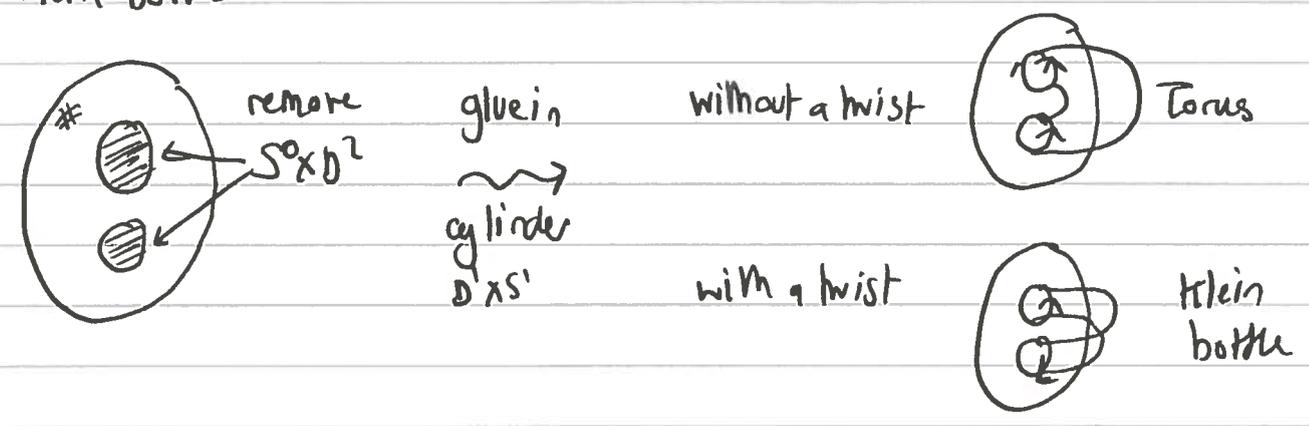
Examples of low dimensional surgeries:

i)  $M = S^1 = \partial(D^2) = \partial(D^1 \times b^1) = S^0 \times b^1 \cup \underbrace{b^1 \times S^0}_{S^0 \times S^0}$

The two possibilities for  $M'$  are either   $\cong S^1 \amalg S^1$   
 or   $\cong S^1$ . Note that surgery can have a drastic effect on connectivity!

In the first case the trace is a pair of pants , in the second case it is a Mobius band with an open disc removed.

ii)  $M = S^2$ . Doing a zero surgery produces a torus or a Klein bottle:



Where do the (framed) embeddings  $S^k \times D^{n-k} \hookrightarrow M^n$  come from?  
 Whitney's embedding theorem: If  $2k < n$  and  $n \geq 5$  then any map  $f: L^k \rightarrow M^n$  can be homotoped to an embedding  $L^k \hookrightarrow M^n$ .  
 Tubular neighbourhood theorem implies that an embedding  $g: S^k \hookrightarrow M^n$  extends to an embedding  $\bar{g}: S^k \times D^{n-k} \hookrightarrow M^n$  iff the normal bundle  $\nu_g: S^k \rightarrow \text{Bord}(n-k)$  is trivial.

What is the purpose of surgery theory? To decide if a manifold can be simplified (in some sense - e.g. made null cobordant) by a sequence of surgeries.

STEP 0: Make  $M$  0-connected.

Do 0-surgeries on the connected components, which is the same as taking connected sums of the connected components.

STEP 1: Make a 0-connected manifold 1-connected by  $t$ -surgeries. (if possible!). If  $g \neq 0 \in \pi_1(M)$  can be represented by a framed embedding  $\bar{g}: S^1 \times D^{n-1} \hookrightarrow M$  then the effect of the surgery is to kill  $g \in \pi_1(M)$ , i.e.  $M' = (M - S^1 \times D^{n-1}) \cup_{\partial(S^1 \times S^{n-2})} S^1 \times S^{n-2}$  has  $\pi_1(M') = \pi_1(M)$  (providing  $n \geq 4$ ).  $\langle g \rangle$

We cannot always do this: take  $M = \mathbb{R}P^2$ . Then  $g \neq 0 \in \mathbb{Z}_2 = \pi_1(\mathbb{R}P^2)$  can be represented by  $S^1 = \mathbb{R}P^1 \subseteq \mathbb{R}P^2$  but the normal 1-plane bundle  $\nu^1: S^1 \rightarrow \text{Bord}(1)$  is not trivial. For  $\pi_2$ , the element  $1 \in \mathbb{Z} = \pi_2(\mathbb{C}P^2)$  can be represented by  $S^2 = \mathbb{C}P^1 \subseteq \mathbb{C}P^2$  but the normal bundle  $\nu^2$  is the Hopf bundle which is not trivial.  $S^2 \subseteq \mathbb{C}P^2$

See Milnor's paper: A procedure for killing homotopy groups of spheres.

### Algebraic Surgery:

Definition: An  $n$ -dimensional symmetric complex  $(C, \phi)$  over  $\mathbb{Z}$  is an  $n$ -dimensional f.g. free  $\mathbb{Z}$ -module chain complex  $C: C_n \rightarrow C_{n-1} \rightarrow \dots \rightarrow C_1 \rightarrow C_0$  with a chain map  $\phi_0: C^{n-*} \rightarrow C_*$ ,

$$\begin{array}{ccccccc} C_0 & \longrightarrow & C_1 & \longrightarrow & C_2 & \longrightarrow & \dots \longrightarrow C_n \\ \downarrow \phi_0 & & \downarrow \phi_0 & & \downarrow \phi_0 & & \downarrow \phi_0 \\ C_n & \longrightarrow & C_{n-1} & \longrightarrow & C_{n-2} & \longrightarrow & \dots \longrightarrow C_0 \end{array}$$

with higher homotopies:  $\phi_1: \phi_0 \simeq T\phi_0$   
 $\phi_2: \phi_1 \simeq T\phi_1$   
 $\vdots$   
 $\phi_n: \phi_{n-1} \simeq T\phi_{n-1}$

Here  $T: \text{Hom}(\mathbb{C}P, \mathbb{C}Q) \xrightarrow{\cong} \text{Hom}(\mathbb{C}Q, \mathbb{C}P)$   
 $\phi \longmapsto (-1)^{pq} \phi^*$

and  $\{\phi_s: \mathbb{C}^r \rightarrow \mathbb{C}^{n-r+2s}\}_{s \geq 0}$  satisfies  $\phi_{-1} = 0$  and  $d\phi_s + \phi_s d^* + \phi_{s-1} + T\phi_{s-1} = 0$ . We say that  $\phi$  is a symmetric structure on  $\mathbb{C}$ .  
 $\{\phi_s\}_{s \geq -1}$

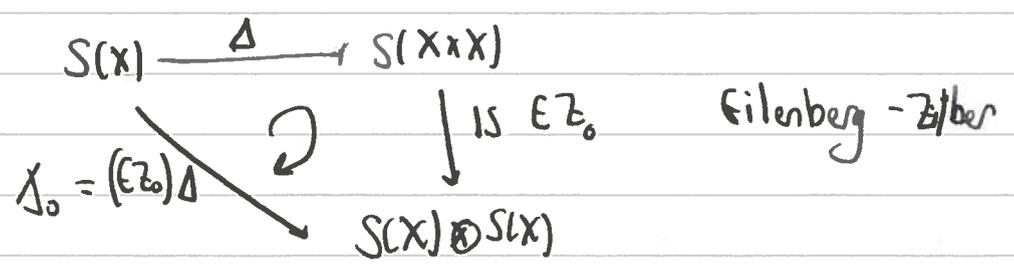
Where do symmetric structures come from? Any space  $X$  with an Alexander-Whitney-Steenrod diagonal chain approximation

$$\Delta = \{\Delta_s | s \geq 0\} : S(X) \longrightarrow S(X) \otimes S(X)$$

$\cong$   
 $\uparrow$   
 $T$

$$T(x \otimes y) = \pm y \otimes x$$

Diagonal map  $\Delta: X \longrightarrow X \times X$   
 $T(x, y) = (y, x)$



$$d\Delta_s + \Delta_s d + \Delta_{s-1} + T\Delta_{s-1} = 0 \text{ for } s \geq 0$$

$$\Delta_{-1} = 0.$$

If  $X$  is a finite  $n$ -dimensional CW complex and  $[X] \in H_n(\mathbb{Z})$  is a homology class then  $(C(X), \{d_s [X] | s \geq 1\}) = (C(X), \phi(X))$  is an  $n$ -dimensional symmetric complex with  $\phi_0 = [X] \cap : C(X)^{n-*} \rightarrow C(X)_*$ .

Definition An  $n$ -dimensional symmetric complex  $(C, \phi)$  is Poincaré if  $\varphi_0 : C^{n-*} \rightarrow C_*$  is a chain equivalence.

Proposition: Let  $X$  be a finite  $n$ -dimensional CW complex and  $[X] \in H_n(\mathbb{Z})$ . Then  $(C(X), \phi(X))$  is an  $n$ -dimensional symmetric Poincaré complex iff  $[X] \cap : H^{n-*}(X) \cong H_n(X)$  is an isomorphism.