

Algebraic Surgery II

Recall that a geometric surgery on an n -dimensional manifold M^n requires :

(i) an embedded sphere $S^k \subset M^n$

(ii) an extension of the embedding $S^k \times D^{n-k} \hookrightarrow M^n$.

Then the effect of the surgery is $M' := \overline{M - S^k \times D^{n-k}} \cup D^{k+1} \times S^{n-k}$

Where do (i) and (ii) come from in topology ?

(i) from Hurewicz and Whitney

Hurewicz: If $\pi_k(M) = 0$ for $0 \leq k \leq r-1$ then $\pi_r(M) \cong H_r(M)$ then every $x \in H_r(M)$ is represented by a map $x: S^r \rightarrow M$

Whitney: If $2r \leq n$ and $n \geq 4$ then we can represent x by an embedding $x: S^r \hookrightarrow M$.

(ii) from Whitehead, Steenrod, Hopf and Whitney. $S^r \hookrightarrow M^n$ has a normal bundle $v: S^r \rightarrow BO(n-r)$. An extension from $S^r \hookrightarrow M^n$ to $S^r \times D^{n-r} \hookrightarrow M^n$ is the same as a trivialisation of the normal $(n-r)$ -plane bundle. An extension exists if and only if $v: S^r \rightarrow BO(n-r)$ is null-homotopic.

If $2r \leq n$ then $[S^r, BO(n-r)] \cong [S^r, BO] \cong \mathbb{Z}_2$ or \mathbb{Z}_2 (Bott)

We don't need an actual trivialisation $v \cong \varepsilon^{n-r}$, it suffices

to have stable trivialisations $v \oplus \varepsilon^k \cong \varepsilon^{n+r+k}$ (Milnor, Kervaire)

In general we cannot kill $\pi_*(M)$ by surgery! (Because cobordism = surgery and not all cobordism groups are zero). \mathbb{RP}^2 is not null-cobordant

and $\mathbb{RP}^1 = S^1 \subset \mathbb{RP}^2$ has (stably) non-trivial normal bundle $v: S^1 \rightarrow BO(1) = \mathbb{RP}^\infty$

and $v \neq 1 \in \pi_1(BO(1)) = \mathbb{Z} \cong \mathbb{Z}_2$. \mathbb{CP}^2 is not null-cobordant and

$\mathbb{CP}^1 = S^2 \subset \mathbb{CP}^2$ and $v: S^2 = \mathbb{CP}^1 \subset \mathbb{CP}^2 \subset BSO(2) = \mathbb{CP}^\infty$ has

non-trivial first Chern class.

Surgery on Normal Maps

Definition A normal map $(f, b): M \rightarrow X$ is a map $f: M^n \rightarrow X$ with a vector bundle $\eta: X \rightarrow \text{Bord}^n$ (large) and a bundle map

$$\begin{array}{ccc} \gamma & & \eta \\ M^n \times S^{n+k} & \xrightarrow{b} & \eta \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & X \end{array}$$

The original example is from Milnor (1960)

$$\begin{array}{ccc} \gamma & \longrightarrow & \eta = E(\pi) \times \mathbb{R}^+ \\ M^n \times S^{n+k} & & \text{universal} \\ \downarrow & & \downarrow \\ M^n & \xrightarrow{f = \gamma_{M^n \times S^{n+k}}} & X = \text{Bord}^n \\ & & \text{tr-plane} \\ & & \text{bundle} \end{array}$$

Instead of trying to make M null-cobordant by surgeries (not possible for \mathbb{RP}^2 , \mathbb{CP}^2) try to make $f: M \rightarrow X$ as close as possible to a homotopy equivalence by surgeries on M keeping X fixed (or at least r -connected for $2r < n$).

Another example is from Pontrjagin. Let M be a framed manifold,

$$\gamma_{M^n \times S^{n+k}} \cong \varepsilon^k$$

$$\begin{array}{ccc}
 V & \longrightarrow & E \\
 \downarrow & & \downarrow \\
 M & \xrightarrow{f} & X = \{pt\}
 \end{array}
 \quad \text{framed cobordism} \quad \Omega_*^{\text{fr}} = \pi_*^S$$

In anticipation of geometric surgery \leftrightarrow algebraic surgery in key cases we want a passage from Geometry to Algebra generalising

Manifold (or Poincaré duality space) $M \longmapsto (C(M), \phi)$ = n-dimensional symmetric complex

where $C(M)$ = chain complex over \mathbb{Z}

$$\phi = S\phi_S : C(M)^{n-r+1} \longrightarrow (C(M)_r | r \geq 0, S \geq 0) \quad \phi_{-1} = 0$$

$$\phi_0 = [M]_{n-r} : C(M)^{n-r} \longrightarrow C(M)_r$$

$$\phi_S : \phi_{S-1} \perp \phi_{S-1}^*$$

$$d\phi_S + \phi_S d^* = \phi_{S-1} + \phi_{S-1}^*$$

There exists a natural diagonal chain approximation $T(a \otimes b) = \pm b \otimes a$

$$\Delta_M : C(M) \longrightarrow \text{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, (C(M) \otimes C(M)) \bigvee_{S(EZ_0)} C(M \times M))$$

Gilenberg-Zilber

$$EZ_1 : (EZ_0)T \cong T(EZ_0)$$

$$EZ_2 : (EZ_1)T \cong T(EZ_1)$$

:

$$d(EZ)_S + (EZ)_S d = (EZ)_{S+1} T + T(EZ)_{S-1}$$

W is the standard free $\mathbb{Z}[\mathbb{Z}_2]$ -resolution

$$W: \dots \rightarrow W_S = \mathbb{Z}[\mathbb{Z}_2] \xrightarrow{1 + (-)^S T} W_{S-1} = \mathbb{Z}[\mathbb{Z}_2] \rightarrow \dots \rightarrow W_0 = \mathbb{Z}[\mathbb{Z}_2]$$

$$(s_M)_S: C(M) \xrightarrow{\Delta} C(M \times M) \xrightarrow{\cong} C(M) \otimes C(M)$$

$\in \mathbb{Z}_S$

If $f: M \rightarrow N$ then there is a commutative diagram

$$\begin{array}{ccc} C(M) & \xrightarrow{\delta_M} & \text{Hom}(W, C(M) \otimes C(M)) \\ f \downarrow & & \downarrow f \otimes f \\ C(N) & \xrightarrow{\delta_N} & \text{Hom}(W, C(N) \otimes C(N)) \end{array}$$

The Steenrod squares $Sq^r: H^{n-r}(M; \mathbb{Z}_2) \rightarrow H^n(M; \mathbb{Z}_2)$

$$(c: C(M) \rightarrow \mathbb{Z}) \mapsto (c(M) \xrightarrow{\Delta} (C(M) \otimes C(M)) \xrightarrow{\text{coev}} \mathbb{Z}_2)_{n-2r} \quad 2n-2r$$

If M^n is an n -manifold with fundamental class $[M] \in H_n(M; \mathbb{Z}_2)$
then the composite

$$H^{n-r}(M; \mathbb{Z}_2) \xrightarrow{Sq^r} H^n(M; \mathbb{Z}_2) \xrightarrow{\langle [M], - \rangle} \mathbb{Z}_2$$

depends only on $\gamma: M \rightarrow \text{Bock}(r)$ (r large). It is
given by the Wu class $V_r(\gamma_M) \in H^r(M; \mathbb{Z}_2) \cong \text{Hom}(H_r(M; \mathbb{Z}_2), \mathbb{Z}_2)$
(theorem of Wu and Thom).

$$H^{n-r}(M; \mathbb{Z}_2) \xrightarrow{Sq^r} H^n(M; \mathbb{Z}_2) \xrightarrow{\langle [M], - \rangle} \mathbb{Z}_2$$

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$$H_r(M; \mathbb{Z}_2) \xrightarrow{\quad} V_r(\gamma_M)$$

A framed n -manifold $(M^n, \nu \cong \xi^k)$ is the same as a normal map $(f, h): M \times_{S^{n+k}} S^{n+k} \rightarrow X$ where $X = \text{Spt}3$ (since there's only one vector bundle over a point).

We can in fact kill $\pi_r(M)$ for $2r < n$ by surgery below the middle dimension. In general, given a normal map $(f, h): M \rightarrow X$ we will be able to do the same but kill the homotopy groups $\pi_{r+1}(f)$ for $2r < n$ (part of the exact sequence $\dots \rightarrow \pi_{r+1}(f) \rightarrow \pi_r(M) \xrightarrow{\delta} \pi_r(X) \rightarrow \dots$). There is a Huillet map $\pi_{r+1}(f) \rightarrow H_{r+1}(f)$ and $H_{r+1}(f)$ behaves like a framed manifold in algebra.

How does a vector bundle $\eta: X \rightarrow \text{Bord}$ (or spherical fibration) translate into cohomology? (Use the Thom space and the Thom class!)

Thom class $U \in \widetilde{H}^k(T(\eta))$ with $\begin{cases} \mathbb{Z}_2\text{-coefficients in general} \\ \mathbb{Z}\text{-coefficients in orientable case} \end{cases}$

$$\begin{array}{ccc} \mathbb{R}^k & \longrightarrow & E(\eta) \longrightarrow X \\ \parallel & | & \downarrow \\ & & \end{array} \quad E(\eta) = \frac{D(\eta)}{S(\eta)} = [E(\eta)]^\infty \text{ one-point compactification}$$

$$\mathbb{R}^k \longrightarrow E(\xi^k) \longrightarrow S^1 \quad T(\xi^k) = [\mathbb{R}^k]^\infty = S^1$$

$$U_\xi = 1 \in \widetilde{H}^k(T(\xi^k)) = \widetilde{H}^k(S^1) = \mathbb{Z}_2 = \{0, 1\}$$

The S-duality theorem (Spanier): If Y is a pointed finite CW complex and $N \geq 0$ is sufficiently large (e.g. $Y \subset S^N$) there exists a pointed finite CW complex Z with $\widetilde{H}^*(Y) \cong \widetilde{H}_{N-k}(Z)$. where $Y_+ = Y \cup \text{Spt}3$

Proof: Given $Y \subset S^N$ (let $(W, \partial W)$ be a regular neighbourhood in S^N ($W \xrightarrow{\sim} Y$ is a simple homotopy equivalence) such that

$$S^N \xrightarrow{\frac{S^N}{S^N - W}} = \frac{W}{\partial W} \xrightarrow{\quad} \frac{W \times W}{W \times \partial W} = W + \Lambda \frac{W}{\partial W}$$

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$$\text{and } H^*(Y) \cong \tilde{H}^*(Y_+) \\ \cong H_{N-Y}(Z)$$

If Y were a manifold then $Z = W/\partial W = T(Y_{|CS^N})$ i.e. the S -dual of a manifold is the Thom space of its normal bundle.