Algebraic Surgery III

The role of the (normal) bundle and S-duality.

These are essential to translate the geometric surgery into algebra. We will need to develop <u>chain bundles</u>.

Alexander Duality

For any $K \subset \mathbb{R}^N$ there is a map

$$K \times (\mathbb{R}^N \setminus K) \longrightarrow S^{N-1}$$

$$(x, y) \longmapsto \frac{x-y}{\|x-y\|}$$

which induces morphisms (for reasonable spaces)

$$\phi_i : H_i(K) \xrightarrow{\cong} H^{N-i-1}(\mathbb{R}^N \setminus K).$$

Assuming K is a simplicial or CW subcomplex, we have

$$H^{N-1}(C(K) \otimes C(\mathbb{R}^N \setminus K)) = H^{N-1}(C(K \times (\mathbb{R}^N \setminus K)))$$

so that ϕ_i an iso for $i \neq 0, N-1$.

Remark • This is an early example of a construction where the cohomology of one space is the homology of another.

• A modern account (in 'L-theory + Topological Manifolds') uses the "supplement"

Theorem (Whitehead)

Every finite subcomplex $K \subset \mathbb{R}^N$ has a closed regular neighbourhood, a codim-0 submanifold $W \subset \mathbb{R}^N$ with $K \hookrightarrow W$ a (simple) htpy equiv

$$\Rightarrow \mathbb{R}^N = W \cup_{\partial W} cl(\mathbb{R}^N \setminus W)$$

or, equiv $\qquad S^N = W \cup_{\partial W} cl(S^N \setminus W)$

$\square$

Alexander duality $H_*(K) \cong H^{N-1-*}(\mathbb{R}^N \setminus K)$ is closely related to Poincaré Lefschetz duality

$$H^*(W) \cong H_{N-*}(W, \partial W)$$

This duality is true for any (oriented) N-dim mfld w/ ∂ $(W, \partial W)$, but for a regular n'hood of K, can use a cofibration

$$\begin{array}{c} cl(S^N \setminus W) \to S^N \longrightarrow S^N / cl(S^N \setminus W) \cong W/\partial W \\ \wr\wr \\ S^N \setminus K \end{array}$$

$\Rightarrow$ exact sequence

$$H_*(S^N \setminus K) \longrightarrow H_*(S^N) \to H_*(W/\partial W) \longrightarrow H_{*-1}(S^N \setminus K) \to 0$$

$$H_*(S^N) = 0 \quad (* \neq 0, N), \qquad H_*(W/\partial W) \cong H^{N-*}(W) \cong H^{N-*}(K)$$

Hence $H^{N-*}(K) \longrightarrow H_{*-1}(S^N \setminus K)$ an iso in the claimed dimensions.

For any finite complex $K \subset \mathbb{R}^N$, can find an embedded reg n'hood $(W, \partial W) \subset S^N$. The spaces

$$\mathbb{R}^N \setminus K, \; W/\partial W$$

Def'n S-dual

are s.t.

$$H^*(\mathbb{R}^N \setminus K) \cong H_{N-1-*}(K) \qquad (* \neq 0, N-1)$$

Replacing $K \subset \mathbb{R}^N$ by $K \subset \mathbb{R}^N \subset \mathbb{R}^{N+1}$, we get

$$(W, \partial W) \times (D^1 \times S^0) = (W', \partial W')$$
$$= (W \times D^1, \partial W \times D^1 \cup W \times S^0)$$

So that
$$W'/\partial W' = \frac{\partial W}{\partial W} \wedge D^1/S^0 = \Sigma(W/\partial W),$$
the (pointed) suspension.

Hence, the S-dual of a finite CW cx is well defined as
$$\Sigma(S^N \setminus K) \simeq W/\partial W$$
so that the stable htpy class is independent of our choice of W

[Recall $X \simeq_s Y$ if $\exists\, k$ s.t. $\Sigma^k X \simeq \Sigma^k Y$]

Theorem (1961 – Atiyah, generalised by Browder + Wall)

If M^m is a compact m-dim manifold, $M \subset S^N$ (N large) with normal bundle $\nu_M : M \to BO(N-m)$ then the pointed space
$$M_+ := M \cup (pt)$$
is S-dual to the Thom space $Th(\nu_M) = E(\nu_M) \cup \{\infty\}$ with:
$$\begin{array}{ccccc} S^N \longrightarrow & T(\nu_M) & \xrightarrow{\Delta} & M_+ \wedge Th(\nu_M) \\ & \| & & \| \\ & W/\partial W & & W_+ \wedge W/\partial W \end{array}$$

So there is a diagram of isomophisms

$$\begin{tikzcd}
H^{m-*}(M) \arrow[rr, "\text{Poincaré}"] \arrow[d, "\text{Thom}"] & & H_*(M) \arrow[d, "M \simeq W"] \\
\tilde{H}^{N-*}(Th(\nu_M)) \arrow[dr, "Th(\nu_M) \simeq W/\partial W"'] & & \tilde{H}_*(W_+) \\
& H^{N-*}(W/\partial W) \arrow[ur, "\text{Poincaré Lefschetz}"'] &
\end{tikzcd}$$

Chain-level version

Let $(C(M), \Delta[M]) =: (C, \varphi)$ be the m-dim symmetric Poincaré cx of M.

$$\Delta : C(M) \longrightarrow \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, C(M) \otimes_{\mathbb{Z}} C(M))$$

from last time. I.e.

$$\phi_s : C^{m-r+s} \longrightarrow C_r \quad ; \quad d\varphi_s + \varphi_s d^* = \varphi_{s-1} + \varphi_{s-1}^* \qquad s \geq 0, \quad \varphi_{-1} = 0$$

$$\varphi_0 : C^{m-*} \longrightarrow C_* \quad \text{is } [M] \cap - \text{ a quasi-iso.}$$

To what extent does (C, φ) depend on ν_M?

<u>Def'n</u> $\quad Q^m(C) = H_m \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, C \otimes C)$

We define a suspension:

$$(SC)_r := C_{r-1}$$

and

$$J : Q^m(C) \longrightarrow Q^{m+1}(SC)$$

$$\varphi = \{\varphi_s\} \longmapsto J\varphi = \{(J\varphi)_s = \varphi_{s-1} \mid s \geq 0\}.$$

<u>E.g.</u> $\quad \tilde{C}(\Sigma X) = S\tilde{C}(X)$

- $(J\varphi)_0 = 0 \qquad$ "cup products vanish in suspension" (geometrically)

- $$\begin{tikzcd}
\tilde{C}(\Sigma X) \arrow[r, "\simeq"] \arrow[dd, "\Delta_{\Sigma X}"] & S\tilde{C}(X) \arrow[d, "\Delta_X"] \\
& S\,\mathrm{Hom}(W, \dot{C}(X) \otimes \dot{C}(X)) \arrow[dl, "S"] \\
\mathrm{Hom}(W, \dot{C}(\Sigma X) \otimes \dot{C}(\Sigma X)) &
\end{tikzcd}$$

__Theorem__ (R, 1980)

The image of $(C,\varphi) = (C(M), \Delta[M])$ under

$$J : Q^m(C) \to Q^{m+1}(SC) \to \dots \to \hat{Q}^m(C) := \varinjlim_k Q^{m+k}(S^k C)$$

depends only on the Thom space $T(\nu_M)$ and the Thom class of ν_M.

□

~~Idea~~ Given $\nu : X \to BO(k)$ (or $BG(k)$), let

$$(D^k, S^{k-1}) \to (D(\nu), S(\nu)) \to X$$

$$Th(\nu) = D(\nu)/S(\nu), \quad u \in \tilde{H}^k(Th(\nu))$$

Embed $Th(\nu) \subset S^N$ with S-dual $Th(\nu)^*$ (we know it exists!)

$$\tilde{H}^{N-*}(Th(\nu)^*) = \tilde{H}_*(Th(\nu))$$

and

$$\begin{aligned}
\tilde{H}^k(Th(\nu)) \cong \tilde{H}_{N-k}(Th(\nu)^*) &\xrightarrow{\Delta_{T(\nu)^*}} Q^{N-k}(C(Th(\nu)^*)) \\
&\phantom{\xrightarrow{\Delta}}\qquad \cong \\
&\phantom{\xrightarrow{\Delta}}\quad Q^{N-k}(C^{N-k-*}(X)) \\
&\phantom{\xrightarrow{\Delta}}\qquad \| \\
&\phantom{\xrightarrow{\Delta}}\quad \hat{Q}^0(C(X)^{-*})
\end{aligned}$$

Take N so big that we have stabilised

"Wu classes" live here

$\Rightarrow$ commutative diagram

$$\begin{array}{ccc}
[M] \in H_m(M) & \longrightarrow & Q^m(C(M)) \\
\downarrow & & \downarrow J \\
U_\nu \in H^{N-m}(T(\nu)) & & \\
\downarrow & & \\
U_\nu^* \in H_m(T(\nu)^*) & \longrightarrow & \hat{Q}^0(C(M)^{-*})
\end{array}$$

$$J\sigma^*(M) = \hat{\sigma}^*(\nu_M)$$

Generalized formula of Wu and Thom.