The algebraic theory of surgery V

An oriented n-dimensional geometric Poincaré complex X determines an n-dim symmetric Poincaré complex over $\mathbb{Z}\pi, X$

$$(C(\tilde{X}), \varphi = \Delta(X) \in Q^n(C(\tilde{X})))$$

f.g. free left $\mathbb{Z}\pi$-module cellular chain cx of universal cover.

$$H_n(X) \xrightarrow{\Delta} Q^n(C(\tilde{X})) = H_n(\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, C(\tilde{X}) \otimes_{\mathbb{Z}\pi} C(\tilde{X})$$

<u>Question</u> • How much of $\varphi = \Delta(X)$ is determined by the SNF

$$\nu_X : X \to BG \quad ?$$

If $(f,b) : (M, \nu_M) \longrightarrow (X, \nu_X)$ is a normal map from a manifold n-dim.

then

$$\begin{tikzcd}
{[M]} \arrow[d, mapsto] & H_n(M) \arrow[r] \arrow[d, "\cong"] & Q^n(C(\tilde{M})) \arrow[d, "\tilde{f}^{\%} = (\tilde{f} \otimes \tilde{f})"] \\
{[X]} & H_n(X) \arrow[r] & Q^n(C(\tilde{X}))
\end{tikzcd}$$

$\tilde{M}$ is pullback of $\tilde{X}$

$\tilde{M} = f^* \tilde{X}$.

$\tilde{f}$ is a chain htpy surjection so there is

$$\begin{tikzcd}
\tilde{f} : C(\tilde{M}) \arrow[r] & C(\tilde{X}) \arrow[l, bend left, "f^!"]
\end{tikzcd} \qquad f^! \tilde{f} = 1.$$

<u>The Umkehr map</u>

$$f^! : C(\tilde{X}) \xrightarrow[\mathbb{Z}\pi\text{-PD}]{\varphi_0} C(\tilde{X})^{n-*} \xrightarrow{\tilde{f}^*} C(\tilde{M})^{n-*} \xrightarrow[\mathbb{Z}\pi\text{-PD}]{} C(\tilde{M}).$$

$$C(\tilde{M}) \simeq C(\tilde{X}) \oplus \mathrm{cone}(f^!) \quad \text{as } f^! \text{ splits } \tilde{f}$$

from a chain htpy direct sum system

$$C(\tilde{X}) \underset{\tilde{f}}{\overset{f^!}{\rightleftarrows}} C(\tilde{M}) \underset{}{\overset{e}{\rightleftarrows}} \mathrm{cone}(f^!)$$

<u>Def'n</u> The kernel $\mathbb{Z}\pi$-modules of a degree 1 map $f: M \to X$ are

$$K_r(M) := \ker(\tilde{f}_* : H_r(\tilde{M}) \to H_r(\tilde{X})) = H_r(\mathrm{cone}(f^!)).$$

So we want to put a quadratic structure on $\mathrm{cone}(f^!)$ and see if we can kill by surgery.

$$Q^n(C \oplus D) = Q^n(C) \oplus Q^n(D) \oplus H_n(C \otimes_{\mathbb{Z}\pi} D)$$
$$\varphi \mapsto (\varphi_C, \varphi_D, \Gamma)$$

$$\varphi = \begin{pmatrix} \varphi_C & \Gamma \\ \Gamma^* & \varphi_D \end{pmatrix}.$$

$$\Rightarrow Q^n(C(\tilde{M})) = Q^n(C(\tilde{X})) \oplus Q^n(\mathrm{cone}(f^!)) \oplus H_n(C(\tilde{X}) \otimes_{\mathbb{Z}\pi} \mathrm{cone}(f^!))$$

and $\varphi_M = \Delta M = (\Delta_X[X], e^{\%}\varphi_M, 0)$

$$\begin{array}{ccc} C(\tilde{X})^{n-1} & \xrightarrow{\ \simeq 0\ } & \mathrm{cone}(f^!) \\ \| & & \uparrow e \\ C(\tilde{X}) & \xrightarrow{\ f^!\ } & C(\tilde{M}) \end{array}$$

Suppose $n = 2k$, a $(-)^k$-symmetric form (P, ϕ) f.g. free

A $2k$-dimensional symmetric complex (C,φ) with $H_r(C)=0$ for $r\neq k$ is essentially the same as a $(-)^k$-symmetric form.

<u>Recall</u>

hyper coho

$$Q^n(C) = H^n(\mathbb{Z}_2; C\otimes C) = H_n(\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, C\otimes C))$$

Now

<u>Def'n</u>

hyper homol

$$Q_n(C) := H_n(W\otimes_{\mathbb{Z}\mathbb{Z}_2}(C\otimes C)) = H_n(\mathbb{Z}_n; C\otimes C)$$

$$= \left\{ \psi \;\middle|\; \begin{array}{l} \psi_s : C^{n-r-s} \longrightarrow C_r \\ d\psi_s + \psi_s d^* + \psi_{s+1} + \psi_{s+1}^* = 0 \quad s\geq 0. \end{array} \right\}$$

<u>Remark</u> Note that $d\psi_0 + \psi_0 d^* + \psi_1 + \psi_1^* = 0$ up to sign

$\Rightarrow$ ψ_0 not a chain map.

1973: Cobordism of n-dim quadratic Poincaré complexes are the Wall surgery obstruction groups.