

The algebraic theory of surgery V

An oriented n -dimensional geometric Poincaré n -complex determines an n -dim symmetric Poincaré complex over $\mathbb{Z}\pi, X$

$$(C(\tilde{X}), \varphi = \Delta(X) \in Q^n(C(\tilde{X})))$$

f.g. free left $\mathbb{Z}\pi$ -module cellular chain ex of universal cover.

$$H_n(X) \xrightarrow{\Delta} Q^n(C(\tilde{X})) = H_n(\text{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, C(\tilde{X}) \otimes_{\mathbb{Z}\pi} C(\tilde{X})))$$

Question • How much of $\varphi = \Delta(X)$ is determined by the SNF $\nu_X: X \rightarrow BG$?

• If $(f, b): (M, \nu_M) \rightarrow (X, \nu_X)$ is a normal map from a manifold n -dim.

then

$$\begin{array}{ccc} [M] & H_n(M) & \longrightarrow Q^n(C(\tilde{M})) \\ \downarrow & \downarrow \cong & \downarrow \tilde{f}^* = (\tilde{f} \otimes \tilde{f}) \\ [X] & H_n(X) & \longrightarrow Q^n(C(\tilde{X})) \end{array}$$

$\tilde{M}$ is pullback of $\tilde{X}$
 $\tilde{M} = f^* \tilde{X}$

$\tilde{f}$ is a chain htpy surjection so there is

$$\begin{array}{ccc} \tilde{f}: C(\tilde{M}) & \longrightarrow & C(\tilde{X}) \\ & \nwarrow f^! & \\ & & \end{array} \quad f^! \tilde{f} = 1$$

The Umkehr map

$$f^!: C(\tilde{X}) \xrightarrow[\mathbb{Z}\pi\text{-PD}]{\varphi_*} C(\tilde{X})^{n-*} \xrightarrow{\tilde{f}^*} C(\tilde{M})^{n-*} \xrightarrow[\mathbb{Z}\pi\text{-PD}]{} C(\tilde{M}).$$

$$C(\tilde{M}) \cong C(\tilde{X}) \oplus \text{cone}(f^!) \quad \text{as } f^! \text{ splits } \tilde{f}$$

from a chain bpy direct sum system

$$C(\tilde{X}) \begin{array}{c} \xrightarrow{\tilde{f}^!} \\ \xleftarrow{\tilde{f}} \end{array} C(\tilde{M}) \begin{array}{c} \xrightarrow{e} \\ \xleftarrow{} \end{array} \text{cone}(f^!)$$

Def'n The kernel $\mathbb{Z}\pi$ -modules of a degree 1 map $f:M \rightarrow X$ are

$$K_r(M) := \ker(\tilde{f}_* : H_r(\tilde{M}) \rightarrow H_r(\tilde{X})) = H_r(\text{cone}(f^!)).$$

So we want to put a quadratic structure on $\text{cone}(f^!)$ and see if we can kill by surgery.

$$Q^n(C \oplus D) = Q^n(C) \oplus Q^n(D) \oplus H_n(C \otimes_{\mathbb{Z}\pi} D)$$

$$\varphi \mapsto (\varphi_C, \varphi_D, \Gamma)$$

$$\varphi = \begin{pmatrix} \varphi_C & \Gamma \\ \Gamma^* & \varphi_D \end{pmatrix}.$$

$$\Rightarrow Q^n(C(\tilde{M})) = Q^n(C(\tilde{X})) \oplus Q^n(\text{cone}(f^!)) \oplus H_n(C(\tilde{X}) \otimes_{\mathbb{Z}\pi} \text{cone}(f^!))$$

$$\text{and } \varphi_M = \Delta M = (\Delta_X[X], e^* \varphi_M, 0)$$

$$\begin{array}{ccc} C(\tilde{X})^{n-1} & \xrightarrow{\cong} & \text{cone}(f^!) \\ \parallel & & \uparrow e \\ C(\tilde{X}) & \xrightarrow{f^!} & C(\tilde{M}) \end{array}$$

Suppose $n=2k$, a $(-)^k$ -symmetric form (P, ϕ) f.g. free

A $2k$ -dimensional symmetric complex (C, φ) with $H_r(C) = 0$ for $r \neq k$ is essentially the same as a $(-)^k$ -symmetric form.

Recall

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$$Q^n(C) = H^n(\mathbb{Z}_2; C \otimes C) = H_n(\text{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, C \otimes C))$$

Now

Def'n

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$$Q_n(C) := \text{Hom}_{\mathbb{Z}} H_n(W \otimes_{\mathbb{Z}\mathbb{Z}_2} (C \otimes C)) = H_n(\mathbb{Z}_n; C \otimes C)$$

$$= \left\{ \psi \mid \begin{aligned} &\psi_s: C^{n-r-s} \rightarrow C_r \\ &d\psi_s + \psi_s d^* + \psi_{s+1} + \psi_{s+1}^* = 0 \quad s \geq 0. \end{aligned} \right\}$$

Remark Note that $d\psi_0 + \psi_0 d^* + \psi_1 + \psi_1^* = 0$ up to sign
 $\Rightarrow \psi_0$ not a chain map.

1973: Cobordism of n -dim quadratic Poincaré complexes are the Wall surgery obstruction groups.

A 2-dimensional coordinate system is shown. The x-axis is horizontal and the y-axis is vertical. The origin is labeled (0,0). The x-axis is labeled x and the y-axis is labeled y. The axes are labeled with numbers 1 through 10. The grid lines are spaced at intervals of 1 unit.

The point (3,4) is plotted. The point is labeled (3,4). The point is located at the intersection of the vertical line x=3 and the horizontal line y=4. The point is marked with a dot.

The point (4,3) is plotted. The point is labeled (4,3). The point is located at the intersection of the vertical line x=4 and the horizontal line y=3. The point is marked with a dot.

The point (5,2) is plotted. The point is labeled (5,2). The point is located at the intersection of the vertical line x=5 and the horizontal line y=2. The point is marked with a dot.

The point (6,1) is plotted. The point is labeled (6,1). The point is located at the intersection of the vertical line x=6 and the horizontal line y=1. The point is marked with a dot.

The point (7,0) is plotted. The point is labeled (7,0). The point is located at the intersection of the vertical line x=7 and the horizontal line y=0. The point is marked with a dot.

The point (8,-1) is plotted. The point is labeled (8,-1). The point is located at the intersection of the vertical line x=8 and the horizontal line y=-1. The point is marked with a dot.

The point (9,-2) is plotted. The point is labeled (9,-2). The point is located at the intersection of the vertical line x=9 and the horizontal line y=-2. The point is marked with a dot.

The point (10,-3) is plotted. The point is labeled (10,-3). The point is located at the intersection of the vertical line x=10 and the horizontal line y=-3. The point is marked with a dot.