

# The Algebraic Theory of Surgery VI

- topologically:
- (i) Spaces  $X$  with  $[X] \in H_n(X)$
  - (ii) Spherical fibrations  $\nu: X \rightarrow BG$
  - (iii) Normal spaces  $(X, \nu, \rho: S^{n+1} \rightarrow T(\nu)), \rho[S^{n+1}] = [X]$
  - (iv) Poincaré duality spaces  $(X, [X]), \nu = \nu_X = \text{Spivak normal fibration}$
  - (v) Normal maps  $(f, b): M \xrightarrow{\nu_M} X$  of Poincaré duality spaces  
 $f[M] = [X] \in H_n(X)$ ,  $\begin{array}{ccc} M & \xrightarrow{\nu_M} & X \\ \downarrow f & & \downarrow \nu_X \\ M & \xrightarrow{f} & X \end{array}$

Algebraically: chain complexes with extra structure

(i) Symmetric chain complexes  $(C, \varphi \in \mathcal{Q}^n(C) = H_n(\text{Hom}(W, \mathbb{C} \otimes C)))$

(ii) Hyperquadratic chain complex  $(C, \gamma \in \hat{\mathcal{Q}}^0(C^{-*}))$   
 $H_0(\text{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(\hat{W}, \mathbb{C}^{-*} \otimes_{\mathbb{Z}} C^{-*}))$

chain bundle

$$\cong \text{Hom}(H^{2*}(0; \mathbb{Z}_2))$$

(iii) Algebraic normal complex  $(C, \varphi, \gamma, \chi), \varphi_0: C^{n-*} \rightarrow C$

$$\varphi \in \mathcal{Q}^n(C) \xrightarrow{J} \hat{\mathcal{Q}}^n(C) \xleftarrow{\hat{\varphi}_0} \hat{\mathcal{Q}}^n(C^{n-*}) = \hat{\mathcal{Q}}^0(C^{-*}) \ni \gamma$$

Formula of Wu and Thom  $J\varphi - \hat{\varphi}_0(\gamma) = d\chi \in \text{Hom}(\hat{W}, \mathbb{C} \otimes C)_n$

Here  $J: \mathcal{Q}^n(C) \rightarrow \hat{\mathcal{Q}}^n(C)$

$$\mathcal{Q}^n(C) \ni \varphi = \{ \varphi_s \in (C \otimes C)_{n+s} \mid s \geq 0, d\varphi_s = \varphi_{s-1} \pm \varphi_{s+1}, \varphi_{-1} = 0 \}$$



$$\hat{\mathcal{Q}}^n(C) \ni J\varphi = \{ (J\varphi)_s \in (C \otimes C)_{n+s} \mid s \in \mathbb{Z} \}$$

$$(J\varphi)_s = \begin{cases} \varphi_s, & s \geq 0 \\ 0, & s \leq -1 \end{cases}$$

The (part) of  $\mathcal{V}$  which is trivial corresponds to  $\ker(\mathcal{J}: \hat{\Phi}^n(\mathbb{C}) \rightarrow \hat{\Phi}^{n-1}(\mathbb{C}))$ .

Fundamental algebraic exact sequence for: quadratic =  $\mathcal{J} = 0$  with reason

$$\dots \rightarrow \underbrace{\Phi_n''(\mathbb{C})}_{H_n(\text{Hom}(W_{-X}, \mathbb{C}\langle\mathbb{C}\rangle))} \xrightarrow{1+T} \underbrace{\Phi_n''(\mathbb{C})}_{H_n(\text{Hom}(W, \mathbb{C}\langle\mathbb{C}\rangle))} \xrightarrow{\mathcal{J}} \underbrace{\hat{\Phi}^n''(\mathbb{C})}_{H_n(\text{Hom}(\hat{W}, \mathbb{C}\langle\mathbb{C}\rangle))} \rightarrow \Phi_{n-1}(\mathbb{C}) \rightarrow \dots$$

quadratic                  symmetric                  hyperquadratic

which is induced by the short exact sequence of free  $\mathbb{Z}\langle\mathbb{Z}_2\rangle$ -modules

$$\hat{\Phi}^{n+1}(\mathbb{C}) \quad 0 \longrightarrow W_{-X-1} \longrightarrow \hat{W} \xrightarrow{\mathcal{J}} W \longrightarrow 0$$

$$\hat{\Phi}_n(\mathbb{C}) = \{ \gamma_s \in (\mathbb{C}\langle\mathbb{C}\rangle)_{n-s} \mid s \geq 0, d(\gamma_s) = \gamma_{s+1} \pm \gamma_{s+1}^* \}$$

$$\downarrow 1+T$$

$$\Phi^n(\mathbb{C}) = \{ \varphi_s \in (\mathbb{C}\langle\mathbb{C}\rangle)_{n+s} \mid s \geq 0, d\varphi_s = \varphi_{s-1} \pm \varphi_{s-1}^*, \varphi_{-1} = 0 \}$$

$$\downarrow \mathcal{J}$$

$$\hat{\Phi}^n(\mathbb{C}) = \text{Hom}(H^{2n}(\mathbb{C}), \mathbb{Z}_2)$$

$$((1+T)\gamma)_s = \begin{cases} (1+T)\gamma_0 & \text{if } s=0 \\ 0 & \text{if } s \geq 1 \end{cases}$$

Example: Let  $X$  be a  $(n-1)$ -connected  $2n$ -dimensional  $M^{2n}$ .  
 $X = S^{2n}$ ,  $f: M \rightarrow S^{2n}$  degree 1,  $[M, S^{2n}] = H^{2n}(M) = \mathbb{Z}$ .

$\exists b: \mathcal{V}_M \rightarrow \mathcal{V}_X = \varepsilon$  if and only if the stable normal bundle  $\mathcal{V}^{so}: M \rightarrow \text{Bso}$  is trivial.

Then  $b: \mathcal{V}_M \cong f^* \mathcal{V}_X = f^* \varepsilon = \varepsilon_M$  is a fibre homotopy trivialisation of  $\mathcal{V}_M = J \mathcal{V}^{so}: M \rightarrow \text{Bsg}$  (this  $J$  is the  $J$ -homomorphism of Hott)

This corresponds to a quadratic structure  $\varphi$  on  $\varphi = \mathbb{1}(M) \in \mathbb{Q}^{2n}(C(M))$

$$C(M): \mathbb{C}_{2n}(M) = \mathbb{Z} \rightarrow \dots \rightarrow 0 \rightarrow \text{Ch}(M) = \text{Hh}(M) = 0 \rightarrow \dots \rightarrow \text{Co}(M) = \mathbb{Z}$$

$$\downarrow f \qquad \qquad \qquad \parallel$$

$$C(X): \mathbb{C}_{2n}(X) = \mathbb{Z} \rightarrow \dots \rightarrow 0 \rightarrow 0 \rightarrow 0 \rightarrow \dots \rightarrow \text{Co}(X) = \mathbb{Z}$$

$$f^!: C(X) \cong C(X)^{2n-k} \xrightarrow{f^*} C(M)^{2n-k} \cong C(M)$$

$C = \mathcal{P}(f^!)$  algebraic mapping cone

$$C: \dots \rightarrow 0 \xrightarrow{\text{dimension } k+1} \text{Hh}(M) \xrightarrow{\text{dimension } k} 0 \xrightarrow{\text{dimension } k-1} \dots$$

$$\varphi \in \mathbb{Q}^{2n}(C) = \ker(1 - T: \text{Hom}_{\mathbb{Z}}(\text{H}^k(M), \text{H}^k(M)) \rightarrow \mathbb{Z})$$

$$= (-)^k \text{-symmetric forms } \text{H}^k(M) \times \text{H}^k(M) \rightarrow \mathbb{Z}$$

$$T: \text{Hom}_{\mathbb{Z}}(\text{H}^k(M), \text{H}^k(M)) \rightarrow \mathbb{Z}, \quad T(\theta) = (-)^k \theta$$

$$\varphi: \text{H}^k(M) \times \text{H}^k(M) \rightarrow \mathbb{Z}$$

$$(x, y) \mapsto \langle x, y, [M] \rangle$$

$$\begin{array}{ccccc}
 \mathbb{Q}_{2k}(\mathbb{C}) & \xrightarrow{1+T} & \mathbb{Q}^{2k}(\mathbb{C}) & \xrightarrow{J} & \hat{\mathbb{Q}}^{2k}(\mathbb{C}) \\
 \parallel & & \parallel & & \parallel \\
 \text{coker} \begin{pmatrix} 1 - (-)^k T : \\ H_k(\mathbb{C}) \otimes H_k(\mathbb{C})^{\oplus} \end{pmatrix} & & \text{ker} \begin{pmatrix} 1 - (-)^k T : \\ H_k(\mathbb{C}) \otimes H_k(\mathbb{C})^{\oplus} \end{pmatrix} & & \frac{\text{ker}(1 - (-)^k T)}{\text{Im}(1 - (-)^k T)} \\
 x \otimes y \mapsto x \otimes y - (-)^k y \otimes x & & & & 
 \end{array}$$

$$\gamma \longmapsto \gamma + (-)^k T \gamma = \varphi \longmapsto \text{diagonal term}$$

$1+T: \mathbb{Q}_{2k}(\mathbb{C}) \rightarrow \mathbb{Q}^{2k}(\mathbb{C})$  is injective if  $k$  is even with image the symmetric forms

If  $k$  is odd and  $\mathcal{V}_M$  is fibre homotopy trivialised (by  $b: \mathcal{V}_M \rightarrow \mathcal{V}_X = \varepsilon$ ) the choice of  $b$  determines  $\gamma$  such that  $\varphi = \gamma - T\gamma$ .

Example:  $M = S^1 \times S^1$  there are two choices of  $b$ : one has kerne invariant 0, the other has kerne invariant 1.

$$\begin{aligned}
 M = S^1 \times S^1 & \xrightarrow{f} X = S^2 & H_1(M) &= \mathbb{Z} \oplus \mathbb{Z} \\
 \varphi = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathbb{Q}_2(\mathbb{C}) & = \text{ker}(1+T: H_1(M) \otimes H_1(M)) \\
 & = \text{skew-symmetric forms}
 \end{aligned}$$

$$b: \mathcal{V}_M \rightarrow \mathcal{V}_X = \varepsilon \quad \gamma_{b_0} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \gamma_{b_0} - T\gamma_{b_0} = \varphi$$

$$\gamma_{b_1} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \gamma_{b_1} - T\gamma_{b_1} = \varphi$$

$$(S^1 \times S^1, b_1) = 1 \in \Omega_2^{\text{tr}} = L_2(\mathbb{Z}) = \mathbb{Z}_2$$

For any ring  $A$  with involution, the Wall surgery obstruction groups (1970)  $L_n(A)$  can be expressed (Ranicki 1960) as the cobordism groups  $(C, \gamma \in \mathcal{P}_n(C))$  with  $C: C_n \rightarrow \dots \rightarrow C_0$  f.g. free  $A$ -module chain complex with  $(1+T)\gamma_0: C^{n-x} \rightarrow C$  a chain equivalence.

$(f, b): M^n \rightarrow X^n$  degree 1 normal map of Poincaré complexes has symmetric signature

$$6_* (f, b) \in = (C(f'), \gamma \in \mathcal{P}_n(C(f')))) \in L_n(\mathbb{Z}[\pi_1 X])$$

This is the Wall surgery obstruction of a normal map.  
These groups are 4-periodic.